Zorn's Lemma, Without Ordinals

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Fundamentally, Zorn's lemma is a statement about partially ordered sets and the existence of maximal elements.

Definition
A partially ordered set is a set P equipped with a binary relation $\preceq$ on P with all of the following properties:

- *for all $x \in P$,*
$$x \preceq x;$$
- *for all $x, y \in P$,*
$$\text{if } x \preceq y \text{ and } y \preceq x, \text{ then } x = y;$$
- *for all $x, y, z \in P$,*
$$\text{if } x \preceq y \text{ and } y \preceq z, \text{ then } x \preceq z.$$

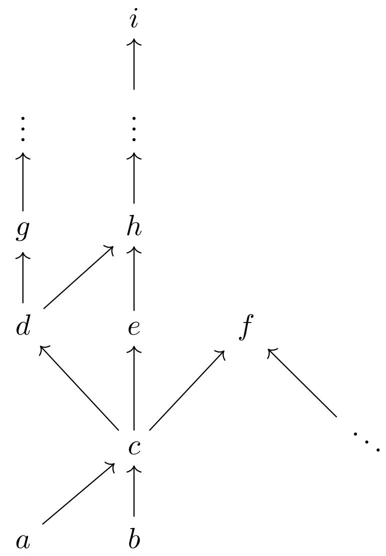
The three properties in the definition above are called reflexivity, antisymmetry, and transitivity respectively.

Given a partially ordered set $(P, \preceq)$, one can construct a strict version of $\preceq$ by declaring

$$x \prec y \text{ if and only if both } x \preceq y \text{ and } x \neq y,$$

for all $x, y \in P$. Note that, in general, the assertion $x \not\prec y$ is *not* equivalent to $x \succeq y$ for partial orders $\preceq$.

One often sees a partial order (intuitively) depicted by a directed graph as below,



where we draw an arrow $x \to y$ to declare the strict relation $x \prec y$.

An upper bound for a subset $S \subseteq P$ is an element $b \in P$ such that

$$s \preceq b \text{ for all } s \in S.$$

For example, in the example depiction above, the set $\{a, b\}$ has many upper bounds: c, e, f, d, among others. As another example, the set $\{a, c, e, h, \dots\}$ has upper bound i. As yet another example, the element e is an upper bound for the set $\{a, b, c, e\}$. In contrast, the set $\{a, c, d, g, \dots\}$ has no upper bound.

A chain in $(P, \preceq)$ is a subset $C \subseteq P$ such that

$$x \preceq y \text{ or } y \preceq x$$

for all $x \in C$. In the example depiction above, the sets $\{a, c, f\}$, $\{b, c, e, h, \dots, i\}$, and $\{c, d\}$ are (not the only) examples of chains. A maximal element of $(P, \preceq)$ is an element $x^* \in P$ such that

$$\text{for all } y \in P, \text{ we have } x \not\prec y.$$

For example, in the example depiction above, f and i are the only maximal elements. Dually, a minimal element of $(P, \preceq)$ is an element $x_* \in P$ such that

$$\text{for all } y \in P, \text{ we have } y \not\prec x.$$

There are two specific kinds of partial orders which are often of interest in order theory. They are the notions of a "linear order" and "well order".

Definition

A linearly ordered set is a partially ordered set $(L, \preceq)$ such that

$$x \preceq y \text{ or } y \preceq x$$

for all $x, y \in L$.

A well ordered set is a linearly ordered set $(W, \preceq)$ such that every non-empty subset $S \subseteq W$ has a minimal element.

Connecting this with the definition of a chain earlier, a partially ordered set $(P, \prec)$ is a linear order if and only if the entire set P itself is a chain in $(P, \prec)$. More concretely, a chain is a linearly ordered subset of a partially ordered set. Linearly ordered sets, as their name suggests, can be intuitively depicted as elements lying on a line.

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet$$

A *well* ordered set, on the other hand, can be thought of as always beginning its construction from the minimum element and building upwards.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

As an example, the set $\mathbb{N}$ of natural numbers equipped with their usual ordering is a well ordered set. The set $\mathbb{R}$ of real numbers equipped with their usual ordering is a linearly ordered set, but *not* a well ordered set.

Well orders allow for induction arguments: if we wanted to prove that a property $\varphi(w)$ holds for all elements w in a well ordered set $(W, \prec)$, then we assume for a contradiction that the set $\{\, w \in W : \neg\varphi(w) \,\}$ is non-empty. The well orderedness of W then yields a minimal element w_0 of this set satisfying $\neg\varphi(w_0)$. We then apply an inductive hypothesis (and we typically use *strong* induction) to conclude that one actually must have had $\varphi(w_0)$, obtaining a contradiction.

We are now in a good position to state Zorn's lemma.

Theorem (Zorn's Lemma)
Let $(P, \preceq)$ be a non-empty partially ordered set such that that every chain $C \subseteq P$ has an upper bound in P. Then $(P, \preceq)$ has a maximal element.

Zorn's lemma often crops up in many fields of mathematics, often in the form of providing proofs for classic results such as:

- every vector space has a (Hamel) basis;
- every nontrivial ring with unity contains a maximal ideal;
- the Hahn-Banach theorem in functional analysis;
- an arbitrary product of compact spaces is compact;

among many other results.

The idea of the proof for Zorn's lemma is fairly straightforward. We pick an element. If this is maximal, then we are done. If not, then we pick something strictly above it. Rinse and repeat. If we don't ever reach a maximal element via repeating this process, then we get a chain of elements. This chain has an upper bound. We then repeat the entire process starting from that upper bound. Eventually, we *must* reach a maximal element, as we will otherwise run out of elements.

There are two key concerns in this strategy. First of all, we do not know what it means to "keep repeating until we run out of elements". The usual approach to this will be to use *transfinite* induction, which we will dance around. Secondly, we may have to make *in*finitely many unspecified choices in the process of "picking the next element". If one attempts to make an "infinitely many unspecified choices" argument in the usual first-order logic, one quickly runs into problems. Indeed, every proof has to be finite. So we cannot write down infinitely many "$\exists$" symbols in a proof. Even if we could, that still would not help us if we had to make *uncountably* many choices. It is for this reason that we adopt the infamous *axiom of choice*, one of the axioms of ZFC set theory which causes contention among some mathematicians who hold a Platonist view of mathematics.

Axiom (The Axiom of Choice)
Let $\mathscr{F}$ be a non-empty family of non-empty sets. Then there exists a function $\textbf{choose}\colon \mathscr{F} \to \bigcup_{S\in\mathscr{F}} S$ *such that* $\textbf{choose}(S) \in S$ *for all* $S \in \mathcal{F}$.

Such a function $\textbf{choose}\colon \mathscr{F} \to \bigcup_{S\in\mathscr{F}} S$ is typically called a <u>choice function for $\mathscr{F}$</u>. Intuitively, this function "chooses" an element from each non-empty set of the family $\mathscr{F}$.

It is only with the aid of this axiom that we may embark on a proof of Zorn's lemma, which we now present.

Proof of Zorn's lemma. We denote by $\mathscr{P}(P)$ the power set of P. There is a choice function $\textbf{choose}\colon \mathscr{P}(P) \setminus \{\varnothing\} \to P$, by virtue of the axiom of choice.

Consider the following sets:

$$\{\text{upper bounds for } S\} := \{\, b \in P \,:\, s \preceq b \text{ for all } s \in S \,\}, \text{ for } S \subseteq P,$$
$$\{\text{strict upper bounds for } S\} := \left\{\, \hat{b} \in P \,:\, s \prec \hat{b} \text{ for all } s \in S \,\right\}, \text{ for } S \subseteq P.$$

Define a function $f\colon \mathscr{P}(P) \to P$ as follows:

$$f(S) := \begin{cases} \textbf{choose}(\{\text{strict upper bounds for } S\}) & \text{if } \{\text{strict upper bounds for } S\} \neq \varnothing, \\ \textbf{choose}(\{\text{upper bounds for } S\}) & \text{if } \{\text{strict upper bounds for } S\} = \varnothing \\ & \quad \text{and } \{\text{upper bounds for } S\} \neq \varnothing, \\ \textbf{choose}(P) & \text{otherwise.} \end{cases}$$

The "otherwise" case in the definition of the function f above is unimportant. Any function satisfying the first two cases will suffice for the proof.

Let $\mathscr{W}$ be the set of all pairs $(W, \trianglelefteq)$ satisfying all of the following properties:

- $(W, \trianglelefteq)$ is a well ordered set, with $W \subseteq P$;
- for all $w \in W$, we have $f(\{\, v \in W \,:\, v \triangleleft w \,\}) = w$.

Note that $\mathscr{W}$ is non-empty. Indeed, the singleton subset $\{f(\varnothing)\} \subseteq P$ equipped with the trivial partial ordering on $\{f(\varnothing)\}$ is a member of $\mathscr{W}$.

If $(V, \sqsubseteq)$ and $(W, \trianglelefteq)$ are two distinct elements of $\mathscr{W}$, then one can prove by induction that:

- either there exists $v_0 \in V$ such that $W = \{\, v \in V \,:\, v \sqsubset w \,\}$;
- or there exists $w_0 \in W$ such that $V = \{\, w \in W \,:\, w \triangleleft w_0 \,\}$.

Consequently, we can let

$$\mathfrak{W} := \bigcup_{(W,\trianglelefteq)\in\mathscr{W}} W \subseteq P$$

and declare a relation $\leq$ on $\mathfrak{W}$ by

$$\mathfrak{a} \leq \mathfrak{b} \text{ if and only if there exists } (W, \trianglelefteq) \in \mathscr{W} \text{ with } \mathfrak{a}, \mathfrak{b} \in W \text{ and } \mathfrak{a} \trianglelefteq \mathfrak{b}$$

to obtain a well ordered set $(\mathfrak{W}, \leq)$ satisfying

$$f(\{\mathfrak{a} \in \mathfrak{W} \,:\, \mathfrak{a} < \mathfrak{b}\}) = \mathfrak{b} \quad \text{for all } \mathfrak{b} \in \mathfrak{W}.$$

Furthermore, the definition of $\mathfrak{W}$ yields $f(\mathfrak{W}) \in \mathfrak{W}$, otherwise we could extend $\mathfrak{W}$ to $\mathfrak{W} \cup \{f(\mathfrak{W})\}$.

Now, recalling that $\prec$ was the original partial ordering on P, we can prove, again by induction, that for $\mathfrak{a}, \mathfrak{b} \in \mathfrak{W}$

$$\text{if } \mathfrak{a} < \mathfrak{b} \text{ then } \mathfrak{a} \prec \mathfrak{b}.$$

Indeed, if this was not the case, then because $\mathfrak{W}$ is well ordered by $<$, there exists a least element $\mathfrak{b}$ of $(\mathfrak{W}, \leq)$ and there exists some $\mathfrak{a} < \mathfrak{b}$ such that $\mathfrak{a} \not\prec \mathfrak{b}$. But then the set $\{\, \mathfrak{c} \in \mathfrak{W} \,:\, \mathfrak{c} < \mathfrak{b} \,\}$ is a chain in $(P, \preceq)$, by the minimality of $\mathfrak{b}$, so $\big\{\text{upper bounds for } \{\, \mathfrak{c} \in \mathfrak{W} \,:\, \mathfrak{c} < \mathfrak{b} \,\}\big\}$ is non-empty, by the hypotheses of Zorn's lemma. Thus, the construction of $\mathfrak{W}$ and the definition of f yield

$$\mathfrak{b} = f(\{\, \mathfrak{c} \in \mathfrak{W} \,:\, \mathfrak{c} < \mathfrak{b} \,\}) \in \big\{\text{upper bounds for } \{\, \mathfrak{c} \in \mathfrak{W} \,:\, \mathfrak{c} < \mathfrak{b} \,\}\big\},$$

giving $\mathfrak{a} \prec \mathfrak{b}$, a contradiction.

Therefore $\mathfrak{W}$ is a chain in $(P, \preceq)$, and so $\mathfrak{W}$ has an upper bound. So $f(\mathfrak{W})$ is an upper bound for $\mathfrak{W}$, by definition of f. But also recall from before that $f(\mathfrak{W}) \in \mathfrak{W}$. By definition of f, this means that there are no strict upper bounds for $\mathfrak{W}$. So $f(\mathfrak{W})$ is a maximal element in $(P, \preceq)$. □

Bibliography

Adam Lawrence Epstein. MA3L1 Topics in Set Theory. Lectures for the academic year 2023–2024 at the University of Warwick, 2024.

Thomas Jech. *Set Theory: The Third Millennium Edition, revised and expanded.* Springer-Verlag Berlin Heidelberg, 2003.
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