Solutions to some exercises in coalgebra

Ryan Tay

some date very far into the future, if ever

a work in progress... draft version 15 May 2026

These are my solutions to all the labelled exercises in Jacobs (2017). This document does not stand on its own; it is meant to supplement the book. I have made some changes to some of the exercises where I felt appropriate.

Contents

1 Motivation

1.1 Naturalness of Coalgebraic Representations

Exercise 1.1.1

1. *Prove that the composition operation ; as defined for coalgebras $S \to \{\perp\} \cup S$ is associative, i.e. satisfies $s_1 \,;\, (s_2 \,;\, s_3) = (s_1 \,;\, s_2) \,;\, s_3$, for all statements $s_1, s_2, s_3 \colon S \to \{\perp\} \cup S$.*

   Define a statement $\mathsf{skip} \colon S \to \{\perp\} \cup S$ *which is a unit for composition ; i.e. which satisfies* $(\mathsf{skip} \,;\, s) = s = (s \,;\, \mathsf{skip})$*, for all* $s \colon S \to \{\perp\} \cup S$.

2. *Do the same for ; defined on coalgebras $S \to \{\perp\} \cup S \cup (S \times E)$.*

(In both cases, statements with an associative composition operation and a unit element form a monoid.)

Solution.

1. Recall that the composition operation ; was defined as follows:

$$s \,;\, t \coloneqq \lambda x \in S.\begin{cases} \perp, & \text{if } s(x) = \perp, \\ t(x') & \text{if } s(x) = x' \in S, \end{cases}$$

   for coalgebras $s, t \colon S \to \{\perp\} \cup S$. Fix any three coalgebras $s_1, s_2, s_3 \colon S \to \{\perp\} \cup S$. Then

$$\begin{aligned} s_1 \,;\, (s_2 \,;\, s_3) &= \lambda x \in S.\begin{cases} \perp, & \text{if } s_1(x) = \perp, \\ (s_2 \,;\, s_3)(x'), & \text{if } s_1(x) = x' \in S, \end{cases} \\ &= \lambda x \in S.\begin{cases} \perp, & \text{if either } s_1(x) = \perp, \text{ or both } s_1(x) = x' \in S \text{ and } s_2(x') = \perp, \\ s_3(x''), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = x'' \in S, \end{cases} \\ &= \lambda x \in S.\begin{cases} \perp, & \text{if } (s_1 \,;\, s_2)(x) = \perp, \\ s_3(x''), & \text{if } (s_1 \,;\, s_2)(x) = x'' \in S, \end{cases} \\ &= (s_1 \,;\, s_2) \,;\, s_3. \end{aligned}$$

   So the composition operation ; is associative.

   The coalgebra $\mathsf{skip} \colon S \to \{\perp\} \cup S$ defined by $\mathsf{skip}(x) \coloneqq x$, for all $x \in S$, satisfies $(\mathsf{skip} \,;\, s) = s = (s \,;\, \mathsf{skip})$ for all coalgebras $s \colon S \to \{\perp\} \cup S$.

2. Now we consider the composition operation ; defined as follows:

$$s \,;\, t \coloneqq \lambda x \in S.\begin{cases} \perp, & \text{if } s(x) = \perp, \\ t(x'), & \text{if } s(x) = x' \in S, \\ (x', e), & \text{if } s(x) = (x', e) \in S \times E, \end{cases}$$

   for coalgebras $s, t \colon S \to \{\perp\} \cup S \cup (S \times E)$. Fix any three coalgebras $s_1, s_2, s_3 \colon \{\perp\} \cup S \cup (S \times E)$. Then

$$s_1 \,;\, (s_2 \,; s_3) = \lambda x \in S.\begin{cases} \perp, & \text{if } s_1(x) = \perp, \\ (s_2 \,;\, s_3)(x'), & \text{if } s_1(x) = x' \in S, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases}$$

$$= \lambda x \in S. \begin{cases} \bot, & \text{if either } s_1(x) = \bot, \text{ or both } s_1(x) = x' \in S \text{ and } s_2(x') = \bot, \\ s_3(x''), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = x'' \in S, \\ (x'', e), & \text{if } s_1(x) = x' \in S \text{ and } s_2(x') = (x'', e) \in S \times E, \\ (x', e), & \text{if } s_1(x) = (x', e) \in S \times E, \end{cases}$$

$$= \lambda x \in S. \begin{cases} \bot, & \text{if } (s_1 \,;\, s_2)(x) = \bot, \\ s_3(x''), & \text{if } (s_1 \,;\, s_2)(x) = x'' \in S, \\ (x'', e), & \text{if } (s_1; s_2)(x) = (x'', e) \in S \times E, \end{cases}$$

$$= (s_1 \,;\, s_2) \,;\, s_3.$$

So this composition operation ; is also associative.

Now define the coalgebra $\mathsf{skip}\colon S \to \{\bot\} \cup S \cup (S \times E)$ by $\mathsf{skip}(x) \coloneqq x$, for all $x \in S$. Then we have $(\mathsf{skip}\,;\, s) = s = (s\,;\, \mathsf{skip})$ for all coalgebras $s\colon S \to \{\bot\} \cup S \cup (S \times E)$. □

Exercise 1.1.2

Define also a composition monoid $(\mathsf{skip}, \,;\,)$ *for coalgebras* $S \to \mathcal{P}(S)$.

Solution. For coalgebras $s, t\colon S \to \mathcal{P}(S)$, define

$$s\,;\, t \coloneqq \lambda x \in S. \left(\bigcup_{y \in s(x)} t(y) \right).$$

Then, for coalgebras $s_1, s_2, s_3\colon S \to \mathcal{P}(S)$, we have

$$\begin{aligned} s_1 \,;\, (s_2 \,;\, s_3) &= \lambda x \in S. \left(\bigcup_{y \in s_1(x)} (s_2 \,;\, s_3)(y) \right) \\ &= \lambda x \in S. \left(\bigcup_{y \in s_1(x)} \bigcup_{z \in s_2(y)} s_3(z) \right) \\ &= \lambda x \in S. \left(\bigcup_{z \in (s_1 \,;\, s_2)(x)} s_3(z) \right) \\ &= (s_1 \,;\, s_2) \,;\, s_3. \end{aligned}$$

Furthermore, defining $\mathsf{skip}\colon S \to \mathcal{P}(S)$ by $\mathsf{skip}(x) \coloneqq \{x\}$ for all $x \in S$, we have

$$\begin{aligned} (\mathsf{skip}\,;\, s) &= \lambda x \in S. \left(\bigcup_{y \in \mathsf{skip}(x)} s(y) \right) \\ &= \lambda x \in S. \left(\bigcup_{y \in \{x\}} s(y) \right) \\ &= \lambda x \in S. s(x) \\ &= s \end{aligned}$$

and

$$\begin{aligned}
(s\,;\,\mathsf{skip}) &= \lambda x \in S.\left(\bigcup_{y \in s(x)} \mathsf{skip}(y)\right)\\
&= \lambda x \in S.\left(\bigcup_{y \in s(x)} \{y\}\right)\\
&= \lambda x \in S.s(x)\\
&= s.
\end{aligned}$$

□

1.2 The Power of Coinduction

Exercise 1.2.1

Compute the $\mathsf{nextdec}$*-behaviour of* $\frac{1}{7} \in [0,1)$ *as in Example 1.2.2.*

Solution. We first recall all of the following functions.

1. The final coalgebra $\mathsf{next}\colon \{0,\dots,9\}^\infty \to \{\bot\} \cup \big(\{0,\dots,9\} \times \{0,\dots,9\}^\infty\big)$ is defined by

$$\mathsf{next}(\sigma) := \begin{cases} \bot, & \text{if } \sigma \text{ is the empty sequence,} \\ (d, \sigma'), & \text{if } \sigma \text{ has head } d \in \{0,\dots,9\} \text{ and tail } \sigma' \in \{0,\dots,9\}^\infty, \text{ i.e. } \sigma = d \cdot \sigma', \end{cases}$$

   for all (finite or infinite) sequences $\sigma \in \{0,\dots,9\}^\infty$.

2. The coalgebra $\mathsf{nextdec}\colon [0,1) \to \{\bot\} \cup \big(\{0,\dots,9\} \times [0,1)\big)$ is defined by

$$\mathsf{nextdec}(r) := \begin{cases} \bot, & \text{if } r = 0, \\ (d, 10r - d), & \text{if } d \le 10r < d+1 \text{ and } d \in \{0,\dots,9\}, \end{cases}$$

   for all $r \in [0,1)$.

3. The function $\mathrm{beh}_{\mathsf{nextdec}}\colon [0,1) \to \{0,\dots,9\}^\infty$ is the unique function making

$$\begin{tikzcd} \{\bot\} \cup \big(\{0,\dots,9\} \times [0,1)\big) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})"] & \{\bot\} \cup \big(\{0,\dots,9\} \times \{0,\dots,9\}^\infty\big) \\ {[0,1)} \arrow[u, "\mathsf{nextdec}"] \arrow[r, dashed, "\exists!\mathrm{beh}_{\mathsf{nextdec}}"'] & \{0,\dots,9\}^\infty \arrow[u, "\cong"', "\mathsf{next}"] \end{tikzcd}$$

   commute.

We wish to compute $\mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right)$. We see that

$$\begin{aligned} \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right) &= \mathsf{next}^{-1}\left(\left(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\right)\left(\mathsf{nextdec}\left(\frac{1}{7}\right)\right)\right) \\ &= \mathsf{next}^{-1}\left(\left(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\right)\left(\left(1, \frac{3}{7}\right)\right)\right) \\ &= \mathsf{next}^{-1}\left(\left(1, \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{3}{7}\right)\right)\right) \\ &= 1 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{3}{7}\right). \end{aligned}$$

Continuing in this fashion,

$$\begin{aligned} \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right) &= 1 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{3}{7}\right) \\ &= 1 \cdot \left(4 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{2}{7}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= 1 \cdot \left(4 \cdot \left(2 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{6}{7}\right)\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{4}{7}\right)\right)\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{5}{7}\right)\right)\right)\right)\right) \\
&= 1 \cdot \left(4 \cdot \left(2 \cdot \left(8 \cdot \left(5 \cdot \left(7 \cdot \mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right)\right)\right)\right)\right)\right).
\end{aligned}
$$

Therefore $\mathrm{beh}_{\mathsf{nextdec}}\left(\frac{1}{7}\right) = \langle 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, 1, 4, 2, 8, 5, 7, \dots \rangle$.

$\square$

Exercise 1.2.2

Formulate appropriate rules for the function $\mathsf{odds}\colon A^\infty \to A^\infty$ *in analogy with the rules (1.7) for* evens.

Solution. We recall that, for a sequence $\sigma \coloneqq \langle a_0, a_1, a_2, a_3, \dots \rangle \in A^\infty$, the function odds satisfies $\mathsf{odds}(\sigma) = \langle a_1, a_3, a_5, \dots \rangle$, and analogously if σ is a finite sequence. The rules we want odds to satisfy are:

$$\frac{\sigma \not\to}{\mathsf{odds}(\sigma) \not\to}$$

i.e. odds should send the empty sequence to the empty sequence;

$$\frac{\sigma \xrightarrow{a} \sigma' \qquad \sigma' \not\to}{\mathsf{odds}(\sigma) \not\to}$$

i.e. odds should send a singleton sequence $\langle a \rangle$ to the empty sequence; and

$$\frac{\sigma \xrightarrow{a} \sigma' \qquad \sigma' \xrightarrow{a'} \sigma''}{\mathsf{odds}(\sigma) \xrightarrow{a'} \mathsf{odds}(\sigma')}$$

i.e. if $\sigma = a \cdot a' \cdot \sigma' \in A^\infty$, where $a, a' \in A$, then $\mathsf{odds}(\sigma) = a' \cdot \mathsf{odds}(\sigma')$. $\square$

Exercise 1.2.3

Use coinduction to define the empty sequence $\langle\rangle \in A^\infty$ *as a map* $\{\bot\} \to A^\infty$.

Fix an element $a \in A$, *and similarly define the infinite sequence* $\overrightarrow{a}\colon \{\bot\} \to A^\infty$ *consisting of only* a*s.*

Solution. We recall that the final coalgebra $\mathsf{next}\colon A^\infty \to \{\bot\} \cup \big(A \times A^\infty\big)$ is defined by

$$
\mathsf{next}(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma \text{ is the empty sequence,} \\ (a, \sigma'), & \text{if } \sigma \text{ has head } a \in A \text{ and tail } \sigma' \in A^\infty \text{, i.e. } \sigma = a \cdot \sigma', \end{cases}
$$

for all (finite or infinite) sequences $\sigma \in A^\infty$.

For the coalgebra $\kappa_1\colon \{\bot\} \to \{\bot\} \cup (A \times \{\bot\})$ defined by $\kappa_1(\bot) \coloneqq \bot$, the unique function $\mathrm{beh}_{\kappa_1}\colon \{\bot\} \to A^\infty$ making

$$
\begin{tikzcd}
\{\bot\} \cup (A \times \{\bot\}) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathrm{beh}_{\kappa_1})"] & \{\bot\} \cup (A \times A^\infty) \\
\{\bot\} \arrow[u, "\kappa_1"] \arrow[r, dashed, "\exists ! \mathrm{beh}_{\kappa_1}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}
$$

commute satisfies $\mathrm{beh}_{\kappa_1}(\bot) = \langle\rangle$.

For the coalgebra $c_a\colon \{\bot\} \to \{\bot\} \cup (A \times \{\bot\})$ defined by $c_a(\bot) \coloneqq (a, \bot)$, the unique function $\mathrm{beh}_{c_a}\colon \{\bot\} \to A^\infty$ making

$$\begin{tikzcd}
\{\bot\} \cup (A \times \{\bot\}) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathrm{beh}_{c_a})"] & \{\bot\} \cup (A \times A^\infty) \\
\{\bot\} \arrow[u, "c_a"] \arrow[r, dashed, "\exists!\mathrm{beh}_{c_a}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute satisfies $\mathrm{beh}_{c_a}(\bot) = \overrightarrow{a} = \langle a, a, a, \dots\rangle$. $\square$

Exercise 1.2.4

Compute the outcome of $\mathsf{merge}(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle)$.

Solution. Recall that we defined the coalgebra $m\colon A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty))$ by

$$m(\sigma, \tau) \coloneqq \begin{cases} \bot, & \text{if } \sigma \not\to \text{ and } \tau \not\to, \\ (a, (\sigma, \tau')), & \text{if } \sigma \not\to \text{ and } \tau \xrightarrow{a} \tau', \\ (a, (\tau, \sigma')), & \text{if } \sigma \xrightarrow{a} \sigma', \end{cases}$$

for all $\sigma, \tau \in A^\infty$, and that $\mathsf{merge}\colon A^\infty \times A^\infty \to A^\infty$ is the unique function making

$$\begin{tikzcd}
\{\bot\} \cup (A \times (A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge})"] & \{\bot\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \arrow[u, "m"] \arrow[r, dashed, "\exists!\mathsf{merge}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute. Then

$$\begin{aligned}
\mathsf{merge}(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle) &= \mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge})\big)\big(m(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle)\big)\Big) \\
&= \mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge})\big)\big((a_0, (\langle b_0, b_1, b_2, b_3\rangle, \langle a_1, a_2\rangle))\big)\Big) \\
&= \mathsf{next}^{-1}\Big(\big(a_0, \mathsf{merge}(\langle b_0, b_1, b_2, b_3\rangle, \langle a_1, a_2\rangle)\big)\Big) \\
&= a_0 \cdot \mathsf{merge}(\langle b_0, b_1, b_2, b_3\rangle, \langle a_1, a_2\rangle),
\end{aligned}$$

and so on. Eventually, we obtain $\mathsf{merge}(\langle a_0, a_1, a_2\rangle, \langle b_0, b_1, b_2, b_3\rangle) = \langle a_0, b_0, a_1, b_1, a_2, b_2, b_3\rangle$. $\square$

Exercise 1.2.5

Is the merge *operation associative, i.e. is* $\mathsf{merge}(\sigma, \mathsf{merge}(\tau, \rho))$ *the same as* $\mathsf{merge}(\mathsf{merge}(\sigma, \tau), \rho)$*? Give a proof or a counterexample. Is there a neutral element for* merge*?*

Solution. The merge operation is not associative:

$$\begin{aligned}
\mathsf{merge}(\langle a\rangle, \mathsf{merge}(\langle b\rangle, \langle c\rangle)) &= \mathsf{merge}(\langle a\rangle, \langle b, c\rangle) \\
&= \langle a, b, c\rangle,
\end{aligned}$$

whereas

$$\begin{aligned}\mathsf{merge}(\mathsf{merge}(\langle a\rangle, \langle b\rangle), \langle c\rangle) &= \mathsf{merge}(\langle a, b\rangle, \langle c\rangle)\\ &= \langle a, c, b\rangle,\end{aligned}$$

for all $a, b, c \in A$.

The neutral element for merge is the empty sequence: for any $\sigma \in A^\infty$, we have $\mathsf{merge}(\sigma, \langle\rangle) = \mathsf{merge}(\langle\rangle, \sigma) = \sigma$. $\square$

Exercise 1.2.6

Show how to define an alternative merge function which alternatingly takes two elements from its argument sequences.

Solution. We will define a coalgebra $m_2\colon A^\infty \times A^\infty \to \{\perp\} \cup (A \times (A^\infty \times A^\infty))$ so that the desired merge function is the unique function $\mathsf{merge}_2\colon A^\infty \times A^\infty \to A^\infty$ making

$$\begin{tikzcd}
\{\perp\} \cup (A \times (A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\perp\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge}_2)"] & \{\perp\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \arrow[u, "m_2"] \arrow[r, dashed, "\exists! \mathsf{merge}_2"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute. As a motivating example, the desired merge of two infinite streams $\langle a_0, a_1, \dots\rangle$ and $\langle b_0, b_1, \dots\rangle$ should be

$$\mathsf{merge}_2(\langle a_0, a_1, a_2, a_3, \dots\rangle, \langle b_0, b_1, b_2, b_3, \dots\rangle) = \langle a_0, a_1, b_0, b_1, a_2, a_3, b_2, b_3, \dots\rangle.$$

As the diagram above commutes, we would require

$$\mathsf{merge}_2\big(m_2(\langle a_0, a_1, a_2, a_3, \dots\rangle, \langle b_0, b_1, b_2, b_3, \dots\rangle)\big) = \big(a_0, \langle a_1, b_0, b_1, a_2, a_3, b_2, b_3, \dots\rangle\big)$$

and so m_2 should be defined to satisfy

$$m_2(\langle a_0, a_1, a_2, a_3, \dots\rangle, \langle b_0, b_1, b_2, b_3, \dots\rangle) = \big(a_0, (\langle a_1, b_0, a_3, b_2, \dots\rangle, \langle b_1, a_2, b_3, a_4, \dots\rangle)\big)$$

Dealing with edge cases separately leads us to the following definition: we define the coalgebra $m_2\colon A^\infty \times A^\infty \to \{\perp\} \cup (A \times (A^\infty \times A^\infty))$ as follows.

1. The function m_2 sends the pair $(\langle\rangle, \langle\rangle)$ to $\perp$, i.e.

$$m_2(\langle\rangle, \langle\rangle) \coloneqq \perp.$$

2. If $\tau \in A^\infty$ is a non-empty sequence, say $\tau \xrightarrow{a} \tau'$ for some $\tau' \in A^\infty$ and $a \in A$, then

$$m_2(\langle\rangle, \tau) \coloneqq \big(a, (\langle\rangle, \tau')\big).$$

3. If $\sigma = \langle a\rangle$ for some $a \in A$, then

$$m_2(\langle a\rangle, \tau) \coloneqq \big(a, (\langle\rangle, \tau)\big)$$

for all $\tau \in A^\infty$.

4. If $\sigma \in A^\infty$ has at least length 2, say $\sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma''$ for some $\sigma', \sigma'' \in A^\infty$ and $a, a' \in A$, then

$$m_2(\sigma, \tau) := \Big(a, \big(\mathsf{merge}(\mathsf{odds}(\sigma), \mathsf{evens}(\tau)), \mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma''))\big)\Big)$$

for all $\tau \in A^\infty$.

Now let $\mathsf{merge}_2 \colon A^\infty \times A^\infty \to A^\infty$ be the unique function which makes

$$\begin{tikzcd}
\{\bot\} \cup (A \times (A^\infty \times A^\infty)) \arrow[rr, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge}_2)"] & & \{\bot\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \arrow[u, "m_2"] \arrow[rr, dashed, "\exists! \mathsf{merge}_2"'] & & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commute. Fix any $\sigma, \tau \in A^\infty$. We argue by cases on (σ, τ) that this function merge_2 is the desired merge function.

1. If $\sigma = \tau = \langle\rangle$, then $\mathsf{merge}_2(\langle\rangle, \langle\rangle) = \langle\rangle$.
2. If $\sigma = \langle\rangle$ and τ is a non-empty sequence, say $\tau = a \cdot \tau'$ for some $a \in A$ and $\tau' \in A^\infty$, then

   $$\mathsf{merge}_2(\langle\rangle, \tau) = a \cdot \mathsf{merge}_2(\langle\rangle, \tau').$$

   Thus $\mathsf{merge}_2(\langle\rangle, \tau) = \tau$.
3. If $\sigma = \langle a \rangle$ for some $a \in A$, then

   $$\begin{aligned} \mathsf{merge}_2(\langle a \rangle, \tau) &= a \cdot \mathsf{merge}_2(\langle\rangle, \tau) \\ &= a \cdot \tau. \end{aligned}$$

4. If $\sigma = a \cdot a' \cdot \sigma''$ for some $a, a' \in A$ and $\sigma'' \in A^\infty$, then

   $$\begin{aligned}
   \mathsf{merge}_2(\sigma, \tau) &= a \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{odds}(\sigma), \mathsf{evens}(\tau)\big), \mathsf{merge}\big(\mathsf{odds}(\tau), \mathsf{evens}(\sigma'')\big)\Big) \\
   &= a \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(a' \cdot \mathsf{odds}(\sigma''), \mathsf{evens}(\tau)\big), \mathsf{merge}\big(\mathsf{odds}(\tau), \mathsf{evens}(\sigma'')\big)\Big) \\
   &= a \cdot a' \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{odds}(\mathsf{merge}(a' \cdot \mathsf{odds}(\sigma''), \mathsf{evens}(\tau))), \\
   &\qquad\qquad \mathsf{evens}(\mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma'')))\big), \\
   &\qquad \mathsf{merge}\big(\mathsf{odds}(\mathsf{merge}(\mathsf{odds}(\tau), \mathsf{evens}(\sigma''))), \\
   &\qquad\qquad \mathsf{odds}(\mathsf{merge}(\mathsf{evens}(\tau), \mathsf{odds}(\sigma'')))\big)\Big) \\
   &= a \cdot a' \cdot \mathsf{merge}_2\Big(\mathsf{merge}\big(\mathsf{evens}(\tau), \mathsf{odds}(\tau)\big), \mathsf{merge}\big(\mathsf{evens}(\sigma''), \mathsf{odds}(\sigma'')\big)\Big) \\
   &= a \cdot a' \cdot \mathsf{merge}_2(\tau, \sigma''),
   \end{aligned}$$

   as desired. $\square$

Exercise 1.2.7

1. *Define three functions* $\mathsf{ex}_i \colon A^\infty \to A^\infty$, *for* $i = 0, 1, 2$, *which extract the elements at positions* $3n + i$.

2. *Define* $\mathsf{merge3}\colon A^\infty \times A^\infty \times A^\infty \to A^\infty$ *satisfying the equation* $\mathsf{merge3}(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)) = \sigma$, *for all* $\sigma \in A^\infty$.

Solution.

1. Define $c_0, c_1, c_2\colon A^\infty \to \{\bot\} \cup (A \times A^\infty)$ as follows:

$$c_0(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle\rangle, \\ (a, \langle\rangle) & \text{if } \sigma = \langle a\rangle \text{ or } \sigma = \langle a, a'\rangle \text{ for some } a, a' \in A, \\ (a, \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases}$$

$$c_1(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle\rangle \text{ or } \sigma = \langle a\rangle \text{ for some } a \in A, \\ (a', \langle\rangle) & \text{if } \sigma = \langle a, a'\rangle \text{ for some } a, a' \in A, \\ (a', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma''', \end{cases}$$

$$c_2(\sigma) := \begin{cases} \bot, & \text{if } \sigma = \langle\rangle, \text{ or } \sigma = \langle a\rangle, \text{ or } \sigma = \langle a, a'\rangle \text{ for some } a, a' \in A, \\ (a'', \sigma'''), & \text{if } \sigma \xrightarrow{a} \sigma' \xrightarrow{a'} \sigma'' \xrightarrow{a''} \sigma'''. \end{cases}$$

Then, for $i \in \{0, 1, 2\}$, the function $\mathsf{ex}_i\colon A^\infty \to A^\infty$ is the unique function making

$$\begin{tikzcd} \{\bot\} \cup (A \times A^\infty) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{ex}_i)"] & \{\bot\} \cup (A \times A^\infty) \\ A^\infty \arrow[u, "c_i"] \arrow[r, dashed, "\exists! \mathsf{ex}_i"'] & A^\infty \arrow[u, "\cong", "\mathsf{next}"'] \end{tikzcd}$$

commute.

2. Define the coalgebra $m_3\colon A^\infty \times A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ by

$$m_3(\sigma, \tau, \rho) := \begin{cases} \bot, & \text{if } \sigma = \tau = \rho = \langle\rangle, \\ \big(a, (\langle\rangle, \langle\rangle, \rho')\big), & \text{if } \sigma = \tau = \langle\rangle \text{ and } \rho \xrightarrow{a} \rho' \text{ for some } a \in A \text{ and } \rho' \in A^\infty, \\ \big(a, (\langle\rangle, \rho, \tau')\big), & \text{if } \sigma = \langle\rangle \text{ and } \tau \xrightarrow{a} \tau' \text{ for some } a \in A \text{ and } \tau' \in A^\infty, \\ \big(a, (\tau, \rho, \sigma')\big), & \text{if } \sigma \xrightarrow{a} \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty. \end{cases}$$

Then we let $\mathsf{merge3}\colon A^\infty \times A^\infty \times A^\infty \to A^\infty$ be the unique function making

$$\begin{tikzcd} \{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge3})"] & \{\bot\} \cup (A \times A^\infty) \\ A^\infty \times A^\infty \times A^\infty \arrow[u, "m_3"] \arrow[r, dashed, "\exists! \mathsf{merge3}"'] & A^\infty \arrow[u, "\cong", "\mathsf{next}"'] \end{tikzcd}$$

commute.

Let us prove that $\mathsf{merge3}(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)) = \sigma$ for all $\sigma \in A^\infty$, by coinduction. Consider the function $f\colon A^\infty \to A^\infty \times A^\infty \times A^\infty$ defined by $f(\sigma) := \big(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)\big)$ for all $\sigma \in A^\infty$.

We wish to show that $\mathsf{merge3} \circ f = \mathrm{id}_{A^\infty}$.

$$\begin{tikzcd}
& \{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) \arrow[r, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times \mathsf{merge3})"] & \{\bot\} \cup (A \times A^\infty) \\
\{\bot\} \cup (A \times A^\infty) \arrow[ur, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times f)"] & A^\infty \times A^\infty \times A^\infty \arrow[u, "m_3"] \arrow[r, "\mathsf{merge3}"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"] \\
A^\infty \arrow[u, "\mathsf{next}", "\cong"'] \arrow[ur, "f"'] & &
\end{tikzcd}$$

Let us first show that the left square commutes. It certainly commutes when we chase the empty sequence: $(m_3 \circ f)(\langle\rangle) = \bot = \big((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f)) \circ \mathsf{next}\big)(\langle\rangle)$. If $\sigma \in A^\infty$ is a non-empty sequence, say $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, then we have

$$\begin{aligned}
(m_3 \circ f)(\sigma) &= m_3\big(\mathsf{ex}_0(\sigma), \mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma)\big) \\
&= \big(a, (\mathsf{ex}_1(\sigma), \mathsf{ex}_2(\sigma), \mathsf{ex}_0(\sigma'))\big) \\
&= \big(a, (\mathsf{ex}_0(\sigma'), \mathsf{ex}_1(\sigma'), \mathsf{ex}_2(\sigma'))\big) \\
&= \big((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times f)) \circ \mathsf{next}\big)(\sigma),
\end{aligned}$$

where the second-to-last equality can also be proven by coinduction. Therefore the outer square commutes, and so

$$\mathsf{next} \circ (\mathsf{merge3} \circ f) = \big((\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times (\mathsf{merge3} \circ f))\big) \circ \mathsf{next}.$$

The finality of the coalgebra $\mathsf{next}\colon A^\infty \to \{\bot\} \cup (A \times A^\infty)$ now yields $\mathsf{merge3} \circ f = \mathrm{id}_{A^\infty}$. $\square$

Exercise 1.2.8

Consider the sequential composition function $\mathsf{comp}\colon A^\infty \times A^\infty \to A^\infty$ *for sequences, described by the three rules:*

$$\frac{\sigma \not\to \qquad \tau \not\to}{\mathsf{comp}(\sigma,\tau) \not\to} \qquad\qquad \frac{\sigma \not\to \qquad \tau \xrightarrow{a} \tau'}{\mathsf{comp}(\sigma,\tau) \xrightarrow{a} \mathsf{comp}(\sigma,\tau')}$$

$$\frac{\sigma \xrightarrow{a} \sigma'}{\mathsf{comp}(\sigma,\tau) \xrightarrow{a} \mathsf{comp}(\sigma',\tau)}.$$

1. *Show by coinduction that the empty sequence* $\langle\rangle = \mathsf{next}^{-1}(\bot) \in A^\infty$ *is a unit element for* comp, *i.e. that* $\mathsf{comp}(\langle\rangle, \sigma) = \sigma = \mathsf{comp}(\sigma, \langle\rangle)$.
2. *Prove also by coinduction that* comp *is associative, and thus that sequences carry a monoid structure.*

Solution.

1. Let $f\colon A^\infty \to A^\infty$ be defined by $f(\sigma) \coloneqq \mathsf{comp}(\langle\rangle, \sigma)$. We will show that the diagram

$$\begin{tikzcd}
\{\bot\} \cup (A \times A^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \,\cup\, (\mathrm{id}_A \times f)"] & \{\bot\} \cup (A \times A^\infty) \\
A^\infty \arrow[u, "\mathsf{next}", "\cong"'] \arrow[r, "f"'] & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

commutes, which would yield $f = \mathrm{id}_{A^\infty}$ by the finality of the coalgebra next.

First, we chase the empty sequence from the bottom left. We see that

$$\begin{aligned}(\mathsf{next} \circ f)(\langle\rangle) &= \mathsf{next}(\mathsf{comp}(\langle\rangle, \langle\rangle)) \\ &= \mathsf{next}(\langle\rangle) \\ &= \perp,\end{aligned}$$

the first rule for comp, and

$$\begin{aligned}\big((\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\langle\rangle) &= \big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big)(\perp) \\ &= \perp.\end{aligned}$$

Now if $\sigma \in A^\infty$ is a non-empty sequence, say $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, we see that

$$\begin{aligned}(\mathsf{next} \circ f)(\sigma) &= \mathsf{next}(\mathsf{comp}(\langle\rangle, a \cdot \sigma')) \\ &= (a, \mathsf{comp}(\langle\rangle, \sigma')) \\ &= (a, f(\sigma')),\end{aligned}$$

by the second rule for comp and the definition of f, and

$$\begin{aligned}\big((\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}\big)(\sigma) &= \big((\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big)\big((a, \sigma')\big) \\ &= (a, f(\sigma')).\end{aligned}$$

Thus $\mathsf{next} \circ f = \big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times f)\big) \circ \mathsf{next}$. This proves that $\mathsf{comp}(\langle\rangle, \sigma) = \sigma$ for all $\sigma \in A^\infty$.

We now show the other equality, that $\mathsf{comp}(\sigma, \langle\rangle) = \sigma$ for all $\sigma \in A^\infty$, we will show that the function $g \colon A^\infty \to A^\infty$ defined by $g(\sigma) \coloneqq \mathsf{comp}(\sigma, \langle\rangle)$ for all $\sigma \in A^\infty$ also satisfies

$$\mathsf{next} \circ g = \big(\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g)\big) \circ \mathsf{next}.$$

That $(\mathsf{next} \circ g)(\perp) = \big((\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g)\big) \circ \mathsf{next}\big)(\perp)$ is the same as with f. Now if $\sigma \in A^\infty$ is such that $\sigma \xrightarrow{a} \sigma'$ for some $a \in A$ and $\sigma' \in A^\infty$, we see that

$$\begin{aligned}(\mathsf{next} \circ g)(\sigma) &= \mathsf{next}(\mathsf{comp}(a \cdot \sigma', \langle\rangle) \\ &= (a, \mathsf{comp}(\sigma', \langle\rangle)) \\ &= (a, g(\sigma')),\end{aligned}$$

by the third rule for comp and the definition of g, and

$$\begin{aligned}\big((\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g)\big) \circ \mathsf{next}\big)(\sigma) &= \big((\mathrm{id}_{\{\perp\}} \cup (\mathrm{id}_A \times g))\big)\big((a, \sigma')\big) \\ &= \big(a, g(\sigma')\big).\end{aligned}$$

Therefore $g = \mathrm{id}_{A^\infty}$, i.e. $\mathsf{comp}(\sigma, \langle\rangle) = \sigma$ for all $\sigma \in A^\infty$.

2. We will define a coalgebra $c \colon A^\infty \times A^\infty \times A^\infty \to \{\perp\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ such that the functions $h, k \colon A^\infty \times A^\infty \times A^\infty \to A^\infty$ given by

$$\begin{aligned}h(\sigma, \tau, \rho) &\coloneqq \mathsf{comp}(\sigma, \mathsf{comp}(\tau, \rho)) \quad \text{and} \\ k(\sigma, \tau, \rho) &\coloneqq \mathsf{comp}(\mathsf{comp}(\sigma, \tau), \rho),\end{aligned}$$

for all $\sigma, \tau, \rho \in A^\infty$, are both coalgebra homomorphisms from c to next.

$$\begin{tikzcd}
\{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty)) \arrow[rr, bend left, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times h)"] \arrow[rr, bend right, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_A \times k)"'] & & \{\bot\} \cup (A \times A^\infty) \\
A^\infty \times A^\infty \times A^\infty \arrow[u, "c"] \arrow[rr, bend left, "h"] \arrow[rr, bend right, "k"'] & & A^\infty \arrow[u, "\cong"', "\mathsf{next}"]
\end{tikzcd}$$

The finality of next would then yield $h = k$.

Define $c \colon A^\infty \times A^\infty \times A^\infty \to \{\bot\} \cup (A \times (A^\infty \times A^\infty \times A^\infty))$ by

$$c(\sigma, \tau, \rho) \coloneqq \begin{cases} \bot, & \text{if } \sigma = \tau = \rho = \langle\rangle, \\ \big(a, (\langle\rangle, \langle\rangle, \rho'\big), & \text{if } \sigma = \tau = \langle\rangle \text{ and } \rho = a \cdot \rho' \text{ for some } a \in A \text{ and } \rho' \in A^\infty, \\ \big(a, (\langle\rangle, \tau', \rho)\big), & \text{if } \sigma = \langle\rangle \text{ and } \tau = a \cdot \tau' \text{ for some } a \in A \text{ and } \tau' \in A^\infty, \\ \big(a, (\sigma', \tau, \rho)\big), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty. \end{cases}$$

Using the rules for comp, it is now elementary to check that h and k make their respective diagrams commute. $\square$

Exercise 1.2.9

Consider two sets A, B with a function $f \colon A \to B$ between them. Use finality to define a function $f^\infty \colon A^\infty \to B^\infty$ that applies f element-wise. Use uniqueness to show that this mapping $f \mapsto f^\infty$ is 'functorial' in the sense that $(\mathrm{id}_A)^\infty = \mathrm{id}_{A^\infty}$ and $(g \circ f)^\infty = g^\infty \circ f^\infty$.

Solution. For a (non-empty) set B, let $\mathsf{next}_B \colon B^\infty \to \{\bot\} \cup (B \times B^\infty)$ denote the final coalgebra defined by

$$\mathsf{next}(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma \text{ is the empty sequence,} \\ (b, \sigma'), & \text{if } \sigma \text{ has head } b \in B \text{ and tail } \sigma' \in B^\infty, \text{ i.e. } \sigma = b \cdot \sigma', \end{cases}$$

for all $\sigma \in B^\infty$. For a function $f \colon A \to B$, define a coalgebra $c_f \colon A^\infty \to \{\bot\} \cup (B \times A^\infty)$ by

$$c_f(\sigma) \coloneqq \begin{cases} \bot, & \text{if } \sigma = \langle\rangle, \\ (f(a), \sigma'), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty, \end{cases}$$

for all $\sigma \in A^\infty$. Let $f^\infty \colon A^\infty \to B^\infty$ be the unique function making

$$\begin{tikzcd}
\{\bot\} \cup (B \times A^\infty) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_B \times f^\infty)"] & \{\bot\} \cup (B \times B^\infty) \\
A^\infty \arrow[u, "c_f"] \arrow[r, dashed, "\exists! f^\infty"'] & B^\infty \arrow[u, "\cong"', "\mathsf{next}_B"]
\end{tikzcd}$$

commute. Then $f(\langle a_0, a_1, a_2, a_3, \dots \rangle) = \langle f(a_0), f(a_1), f(a_2), f(a_3), \dots \rangle$ for all $a_0, a_1, a_2, a_3, \dots \in A$, and analogously for finite sequences.

We see that $c_{\mathrm{id}_A} = \mathsf{next}_A$. So $(\mathrm{id}_A)^\infty = \mathrm{id}_{A^\infty}$ by finality of next_A. Furthermore, for functions $f\colon A \to B$ and $g\colon B \to C$, we see that

$$\begin{tikzcd}
\{\bot\} \cup (C \times A^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times f^\infty)"] & \{\bot\} \cup (C \times B^\infty) \\
A^\infty \arrow[u, "c_{g\circ f}"] \arrow[r, "f^\infty"'] & B^\infty \arrow[u, "c_g"']
\end{tikzcd}$$

commutes. Consequently, the outer square in the diagram

$$\begin{tikzcd}
\{\bot\} \cup (C \times A^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times f^\infty)"] & \{\bot\} \cup (C \times B^\infty) \arrow[r, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times g^\infty)"] & \{\bot\} \cup (C \times C^\infty) \\
A^\infty \arrow[u, "c_{g\circ f}"] \arrow[r, "f^\infty"'] & B^\infty \arrow[u, "c_g"'] \arrow[r, "g^\infty"'] & C^\infty \arrow[u, "\cong", "\mathsf{next}_C"']
\end{tikzcd}$$

commutes, i.e.

$$\mathsf{next}_C \circ (g^\infty \circ f^\infty) = \big(\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_C \times (g^\infty \circ f^\infty))\big) \circ c_{g\circ f}.$$

The finality of next_C then yields $(g \circ f)^\infty = g^\infty \circ f^\infty$. □

Exercise 1.2.10

Use finality to define a map $\mathrm{st}\colon A^\infty \times B \to (A \times B)^\infty$ *that maps a sequence* $\sigma \in A^\infty$ *and an element* $b \in B$ *to a new sequence in* $(A \times B)^\infty$ *by adding this* b *at every position in* σ*. (This is an example of a 'strength' map; see Exercise 2.5.4.)*

Solution. Define a coalgebra $c\colon A^\infty \times B \to \{\bot\} \cup \big((A \times B) \times (A^\infty \times B)\big)$ as follows:

$$c(\sigma, b) \coloneqq \begin{cases} \bot, & \text{if } \sigma = \langle\rangle, \\ \big((a, b), (\sigma', b)\big), & \text{if } \sigma = a \cdot \sigma' \text{ for some } a \in A \text{ and } \sigma' \in A^\infty, \end{cases}$$

for all $\sigma \in A^\infty$ and $b \in B$. The unique function $\mathrm{st}\colon A^\infty \times B \to (A \times B)^\infty$ making

$$\begin{tikzcd}
\{\bot\} \cup \big((A \times B) \times (A^\infty \times B)\big) \arrow[r, dashed, "\mathrm{id}_{\{\bot\}} \cup (\mathrm{id}_{A\times B} \times \mathrm{st})"] & \{\bot\} \cup \big((A \times B) \times (A \times B)^\infty\big) \\
A^\infty \times B \arrow[u, "c"] \arrow[r, dashed, "\exists! \mathrm{st}"'] & (A \times B)^\infty \arrow[u, "\cong", "\mathsf{next}"']
\end{tikzcd}$$

commute will satisfy $\mathrm{st}(\langle a_0, a_1, a_2, \dots\rangle, b) = \langle (a_0, b), (a_1, b), (a_2, b), \dots\rangle$ for all $a_0, a_1, a_2, a_3, \dots \in A$ and $b \in B$, and analagously for finite sequences in A^∞. □

1.3 Generality of Temporal Logic of Coalgebras

Exercise 1.3.1
*The nexttime operator $\bigcirc$ introduced in (1.9) is the so-called **weak** nexttime. There is an associated **strong** nexttime, given by $\neg\bigcirc\neg$. Note the difference between weak and strong nexttime for sequences.*

Solution. Recall that, for a sequence coalgebra $c\colon S \to \{\bot\} \cup (A \times S)$ and a predicate $P \subseteq S$, we have

$$(\bigcirc P)(x) \quad \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in A \times P,$$

for all $x \in S$. So,

$$(\bigcirc\neg P)(x) \quad \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in A \times (S \setminus P),$$

and thus

$$(\neg\bigcirc\neg P)(x) \quad \text{if and only if} \quad c(x) \neq \bot \text{ and } c(x) \notin A \times (S \setminus P).$$

Since the codomain of c is $\{\bot\} \cup (A \times S)$, and since $P \subseteq S$, we can equivalently write this as

$$(\neg\bigcirc\neg P)(x) \quad \text{if and only if} \quad c(x) \in A \times P. \qquad \square$$

Exercise 1.3.2
Prove that the 'truth' predicate that always holds is a (sequence) invariant. And if P_1 and P_2 are invariants, then so is the intersection $P_1 \cap P_2$. Finally, if P is an invariant, then so is $\bigcirc P$.

Solution. Fix a sequence coalgebra $c\colon S \to \{\bot\} \cup (A \times S)$. The truth predicate is the set S itself. Then, for all $x \in S$,

$$(\bigcirc S)(x) \quad \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in A \times S.$$

Since the codomain of c is $\{\bot\} \cup (A \times S)$, this means that $\bigcirc S = S$, and so S is an invariant.

Now suppose that P_1 and P_2 and invariant, i.e. $P_1 \subseteq \bigcirc P_1$ and $P_2 \subseteq \bigcirc P_2$. Then, for all $x \in S$,

$$\begin{aligned}
(\bigcirc(P_1 \cap P_2))(x) \quad & \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in A \times (P_1 \cap P_2) \\
& \text{if and only if} \quad c(x) = \bot \text{ or } c(x) \in (A \times P_1) \cap (A \times P_2) \\
& \text{if and only if} \quad \big(c(x) = \bot \text{ or } c(x) \in A \times P_1\big) \text{ and } \big(c(x) = \bot \text{ or } c(x) \in A \times P_2)\big) \\
& \text{if and only if} \quad (\bigcirc P_1)(x) \text{ and } (\bigcirc P_2)(x).
\end{aligned}$$

Hence $P_1 \cap P_2 \subseteq (\bigcirc P_1) \cap (\bigcirc P_2) = \bigcirc(P_1 \cap P_2)$, and so $P_1 \cap P_2$ is also invariant.

Finally, suppose that P is invariant, i.e. $P \subseteq \bigcirc P$. We aim to show that $\bigcirc P \subseteq \bigcirc\bigcirc P$. Suppose $x \in S$ is such that $(\bigcirc P)(x)$ holds. Then either $c(x) = \bot$ or $c(x) \in A \times P \subseteq A \times \bigcirc P$. Therefore $(\bigcirc\bigcirc P)(x)$ holds. $\square$

Exercise 1.3.3

1. *Show that $\Box$ is an interior operator, i.e. satisfies: $\Box P \subseteq P$, $\Box P \subseteq \Box\Box P$, and $P \subseteq Q \implies \Box P \subseteq \Box Q$.*

2. *Prove that a predicate P is invariant if and only if $P = \Box P$.*

Solution. Fix a sequence coalgebra $c\colon S \to \{\bot\} \cup (A \times S)$. Recall that the henceforth operator $\Box$ is defined on predicates $P \subseteq S$ as follows: for all $x \in S$,

$$(\Box P)(x) \quad \text{if and only if} \quad \text{there exists an invariant } Q \subseteq S \text{ with } x \in Q \subseteq P.$$

In other words, $\Box P$ is the union of all invariants contained in P.

1. If $x \in \Box P$, then there is an invariant $Q \subseteq S$ with $x \in Q \subseteq P$. So $x \in P$ too. Also, Q is an invariant with $x \in Q \subseteq \Box P$. So $x \in \Box\Box P$ as well. Thus $\Box P \subseteq P$ and $\Box P \subseteq \Box\Box P$.

   Now suppose $P \subseteq Q \subseteq S$. Then, for any $x \in \Box P$, there is an invariant $R \subseteq S$ with $x \in R \subseteq P \subseteq Q$. So $x \in \Box Q$ as well. Therefore $\Box P \subseteq \Box Q$.

2. For the forward direction, suppose that P is invariant. By definition, $\Box P$ is the union of all invariants contained within P. As P is assumed to be an invariant, we must have $\Box P = P$.

   For the converse direction, suppose that $\Box P = P$. We need to show that P is an invariant, i.e. $P \subseteq \bigcirc P$. For any $x \in P = \Box P$, there exists an invariant $Q \subseteq S$ with $x \in Q \subseteq P$. As Q is an invariant, either $c(x) = \bot$ or $c(x) \in A \times Q \subseteq A \times P$. Hence we also have $x \in \bigcirc P$. Therefore $P \subseteq \bigcirc P$, meaning P is an invariant. □

Exercise 1.3.4
Recall the finite behaviour predicate $\Diamond\big((-) \not\to\big)$ from Example 1.3.4.1 and show that it is an invariant: $\Diamond\big((-) \not\to\big) \subseteq \bigcirc\Diamond\big((-) \not\to\big)$. Hint: For an invariant Q, consider the predicate $Q' = \big(\neg(-) \not\to\big) \cap (\bigcirc Q)$.

Solution. Fix a sequence coalgebra $c\colon S \to \{\bot\} \cup (A \times S)$. Recall that, for a predicate $P \subseteq S$ and $x \in S$,

$$(\Diamond P)(x) \quad \text{if and only if} \quad \text{for all invariants } Q \subseteq S \text{, we have } \neg Q(x) \text{ or } Q \not\subseteq \neg P.$$

That is, $\Diamond P = \neg\Box\neg P$.

Suppose $x \in S$ is such that $\Diamond\big(x \not\to\big)$ holds. We need to show that $\bigcirc\Diamond\big(x \not\to\big)$ holds, i.e. if $x \xrightarrow{a} x'$ for some $(a, x') \in A \times S$, then $\Diamond\big(x' \not\to\big)$ also holds. Fix any invariant $Q \subseteq S$ with $Q \subseteq \neg\big((-) \not\to\big)$. We need to show that $\neg Q(x')$.

Following the hint, we consider the predicate

$$Q' \coloneqq \neg\big((-) \not\to\big) \cap (\bigcirc Q).$$

Observe that Q' is an invariant: if $y \in S$ satisfies $Q'(y)$, then there is some $(b, y') \in A \times S$ such that $y \xrightarrow{b} y'$ and $Q(y')$ hold. Then, since $Q \subseteq \neg\big((-) \not\to\big)$ and Q is an invariant, we conclude that $Q'(y')$ also holds. So $Q' \subseteq \bigcirc Q'$.

Hence if $Q(x')$ holds, then $Q'(x)$ holds too, contradicting the assumption that $\Diamond\big(x \not\to\big)$. □

Exercise 1.3.5
Let $(A, \leq)$ be a complete lattice, i.e. a poset in which each subset $U \subseteq A$ has a join $\bigvee U \in A$. It is well known that each subset $U \subseteq A$ then also has a meet $\bigwedge U \in A$, given by $\bigwedge U = \bigvee\{\, a \in A \mid \forall b \in U.a \leq b \,\}$.

Let $f\colon A \to A$ be a monotone function: $a \leq b$ implies $f(a) \leq f(b)$. Recall, e.g. from Davey and Priestley (1990, Chapter 4) that such a monotone f has both a least fixed point $\mu f \in A$ and a greatest fixed point $\nu f \in A$ given by the formulas:

$$\mu f = \bigwedge\{\, a \in A \mid f(a) \leq a \,\}, \qquad \nu f = \bigvee\{\, a \in A \mid a \leq f(a) \,\}.$$

Now let $c\colon S \to \{\bot\} \cup (A \times A)$ be an arbitrary sequence coalgebra, with associated nexttime operator $\bigcirc$.

1. *Prove that $\bigcirc$ is a monotone function $\mathcal{P}(S) \to \mathcal{P}(S)$, i.e. that $P \subseteq Q$ implies $\bigcirc P \subseteq \bigcirc Q$, for all $P, Q \subseteq S$.*

2. *Check that $\Box P \in \mathcal{P}(S)$ is the greatest fixed point of the function $\mathcal{P}(S) \to \mathcal{P}(S)$ given by $U \mapsto P \cap \bigcirc U$.*

3. *Define for $P, Q \subseteq S$ a new predicate $P\ \mathcal{U}\ Q \subseteq S$, for 'P until Q' as the least fixed point of $U \mapsto Q \cup (P \cap \neg\bigcirc\neg U)$. Check that 'until' is indeed a good name for $P\ \mathcal{U}\ Q$, since it can be described explicitly as*

$$\begin{aligned} P\ \mathcal{U}\ Q = \{\, x \in S \mid\ & \exists n \in \mathbb{N}. \exists x_0, x_1, \ldots, x_n \in S. \\ & x_0 = x \wedge (\forall i < n. \exists a. x_i \xrightarrow{a} x_{i+1}) \wedge Q(x_n) \\ & \wedge \forall i < n. P(x_i) \,\}. \end{aligned}$$

Hint: Don't use the fixed point definition μ, but first show that this subset is a fixed point, and then that it is contained in an arbitrary fixed point.

(The fixed point definitions that we described above are standard in temporal logic; see e.g. Emerson (1990, 3.24–3.25). The above operation $\mathcal{U}$ is what is called the 'strong' until. The 'weak one' does not have the negations $\neg$ in its fixed-point description in point 3.)

Solution.

1. For subsets $P, Q \in \mathcal{P}(S)$ with $P \subseteq Q$, and for $x \in S$ such that $(\bigcirc P)(x)$ holds, we have

$$c(x) = \perp \text{ or } c(x) \in A \times P.$$

From the assumption that $P \subseteq Q$, it follows that

$$c(x) = \perp \text{ or } c(x) \in A \times Q,$$

or equivalently, $(\bigcirc Q)(x)$.

2. Fix $P \in \mathcal{P}(S)$ and define $f_P \colon \mathcal{P}(S) \to \mathcal{P}(S)$ by $f_P(U) \coloneqq P \cap \bigcirc U$ for all $U \in \mathcal{P}(S)$. Then the greatest fixed point of f_P is

$$\nu(f_P) \coloneqq \bigcup_{\substack{U \in \mathcal{P}(S), \\ U \subseteq f_P(U)}} U = \bigcup_{\substack{U \in \mathcal{P}(S), \\ U \subseteq P \cap \bigcirc U}} U = \Box P.$$

3. Fix $P, Q \in \mathcal{P}(S)$, and define $f_{P,Q} \colon \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$f_{P,Q}(U) \coloneqq Q \cup (P \cap \neg\bigcirc\neg U)$$

for all $U \in \mathcal{P}(S)$. Recall, from Exercise 1.3.1, that

$$\neg\bigcirc\neg U = \{\, x \in S \ :\ c(x) \in A \times U \,\}.$$

We wish to show that the set

$$\begin{aligned} U_{P,Q} \coloneqq Q \cup \Big\{\, x \in S \ :\ & \text{there exist } n \in \mathbb{Z}_{>0},\ x_0, \ldots, x_n \in S \text{ and } a_0, \ldots, a_{n-1} \in A \\ & \text{such that } x = x_0 \xrightarrow{a_0} \cdots \xrightarrow{a_{n-1}} x_n \text{ and} \\ & P(x_0), \ldots, P(x_{n-1}), \text{ and } Q(x_n) \text{ all hold} \Big\} \end{aligned}$$

is the least fixed point of $f_{P,Q}$.

First, observe that

$$f_{P,Q}(U_{P,Q}) = Q \cup (P \cap \neg\bigcirc\neg U_{P,Q})$$

$$
\begin{aligned}
&= Q \cup (P \cap \{\, x \in S \;:\; c(x) \in A \times U_{P,Q} \,\}) \\
&= Q \cup \{\, x \in S \;:\; P(x) \text{ and } c(x) \in A \times U_{P,Q} \,\} \\
&= U_{P,Q},
\end{aligned}
$$

so that $U_{P,Q}$ is indeed a fixed point of $f_{P,Q}$.

Now we show that $U_{P,Q}$ is the least fixed point of $f_{P,Q}$. Fix some $B \subseteq S$ with $f_{P,Q}(B) = B$, i.e.

$$
Q \cup \{\, x \in S \;:\; P(x) \text{ and } c(x) \in A \times B \,\} = B.
$$

Then we get $U_{P,Q} \subseteq B$ by induction on the length of finite sequences $x_0, \ldots x_n \in S$ and $a_0, \ldots, a_{n-1} \in A$ satisfying $x_0 \xrightarrow{a_0} \cdots \xrightarrow{a_{n-1}} x_n$, and $P(x_0) \wedge \cdots \wedge P(x_{n-1}) \wedge Q(x_n)$. $\square$

1.4 Abstractness of Coalgebraic Notions

Exercise 1.4.1
Let $(M, +, 0)$ be a monoid, considered as a category. Check that a functor $F\colon M \to \mathbf{Sets}$ can be identified with a ***monoid action****: a set X together with a function $\mu\colon X \times M \to X$ with $\mu(x, 0) = x$ and $\mu(x, m_1 + m_2) = \mu(\mu(x, m_2), m_1)$.*

Solution. Suppose we are given functor $F\colon M \to X$. This F sends the unique object $\star \in \mathsf{Obj}(M)$ to a set $F(\star) \in \mathsf{Obj}(\mathbf{Sets})$, and sends each $m \in \mathsf{Arr}(M)$ to a function $Fm\colon F(\star) \to F(\star)$. The functoriality of F requires that $F(0) = \mathrm{id}_{F(\star)}$ and $F(m_1 + m_2) = F(m_1) \circ F(m_2)$ for all $m_1, m_2 \in \mathsf{Arr}(M)$. We then define a function $\theta_F\colon F(\star) \times \mathsf{Arr}(M) \to F(\star)$ by $\theta_F(x, m) \coloneqq F(m)(x)$ for all $(x, m) \in F(\star) \times M$.

The equality $\theta_F(x, 0) = x$ for all $x \in F(\star)$ follows the equality $F(0) = \mathrm{id}_{F(\star)}$, while the equality $\theta_F(x, m_1 + m_2) = \theta_F(\mu_F(x, m_2), m_1)$ for all $x \in X$ and $m_1, m_2 \in \mathsf{Arr}(M)$ follows from the equality $F(m_1 + m_2) = F(m_1) \circ F(m_2)$.

Now suppose we are given also given a set X and a function $\mu\colon X \times \mathsf{Arr}(M) \to X$ with $\mu(x, 0) = x$ and $\mu(x, m_1 + m_2) = \mu(\mu(x, m_2), m_1)$ for all $x \in X$ and $m, m_1, m_2 \in \mathsf{Arr}(M)$. We then define a functor $G_\mu\colon M \to \mathbf{Sets}$ by $G_\mu(\star) \coloneqq X$, for the unique object $\star \in \mathsf{Obj}(M)$, and $G_\mu(m) \coloneqq \mu(-, m)$ for each $m \in \mathsf{Arr}(M)$. That G_μ is actually a functor follows from the assumptions on μ.

We then have $G_{\theta_F} = F$ and $\theta_{G_\mu} = \mu$. □

Exercise 1.4.2
Check in detail that the opposite $\mathbb{C}^{\mathrm{op}}$ and the product $\mathbb{C} \times \mathbb{D}$ are indeed categories.

Solution. Let $\mathbb{C}$ and $\mathbb{D}$ be categories.

We defined $\mathsf{Obj}(\mathbb{C}^{\mathrm{op}}) \coloneqq \mathsf{Obj}(\mathbb{C})$. For $X, Y \in \mathsf{Obj}(\mathbb{C})$, write $\hom_{\mathbb{C}}(X, Y)$ for the set of all morphisms with domain X and codomain Y. We then defined $\hom_{\mathbb{C}^{\mathrm{op}}}(X, Y) \coloneqq \hom_{\mathbb{C}}(Y, X)$, and we defined a compositition $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\mathbb{C}^{\mathrm{op}}$ to be the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{C}$. The associativity and identity laws for composition in $\mathbb{C}^{\mathrm{op}}$ follow from those for $\mathbb{C}$.

We defined $\mathsf{Obj}(\mathbb{C} \times \mathbb{D}) \coloneqq \mathsf{Obj}(\mathbb{C}) \times \mathsf{Obj}(\mathbb{D})$. For $X, X' \in \mathsf{Obj}(\mathbb{C})$ and $Y, Y' \in \mathsf{Obj}(\mathbb{D})$, we let $\hom_{\mathbb{C}\times\mathbb{D}}((X, Y), (X', Y')) \coloneqq \hom_{\mathbb{C}}(X, X') \times \hom_{\mathbb{D}}(Y, Y')$. A composition $(X, Y) \xrightarrow{(f,g)} (X', Y') \xrightarrow{(f',g')} (X'', Y'')$ in $\mathbb{C} \times \mathbb{D}$ is defined to be the composition $(X, Y) \xrightarrow{(f'f,\, g'g)} (X'', Y'')$. For an object (X, Y) in $\mathbb{C} \times \mathbb{D}$, the identity morphism $\mathrm{id}_{(X,Y)}$ is the pair $(\mathrm{id}_X, \mathrm{id}_Y)$. The associativity and identity laws for composition in $\mathbb{C} \times \mathbb{D}$ follow from those for $\mathbb{C}$ and $\mathbb{D}$. □

Exercise 1.4.3
Assume an arbitrary category $\mathbb{C}$ with an object $I \in \mathbb{C}$. We form a new category $\mathbb{C}/I$, the so-called ***slice category*** *over I, with*

objects *maps $f\colon X \to I$ with codomain I in $\mathbb{C}$*

morphisms *from $X \xrightarrow{f} I$ to $Y \xrightarrow{g} I$ are morphisms $h\colon X \to Y$ in $\mathbb{C}$ for which the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

1. *Describe identities and composition in $\mathbb{C}/I$, and verify that $\mathbb{C}/I$ is a category.*

2. *Check that taking domains yields a functor $\mathrm{dom}\colon \mathbb{C}/I \to \mathbb{C}$.*

3. *Verify that for $\mathbb{C} = \mathbf{Sets}$, a map $f\colon X \to I$ may be identified with an I-indexed family of sets $(X_i)_{i\in I}$, namely where $X_i = f^{-1}(i)$. What do morphisms in $\mathbb{C}/I$ correspond to, in terms of such indexed families?*

Solution.

1. The identities and composition in $\mathbb{C}/I$ are simply the identities and composition in $\mathbb{C}$. So the fact that $\mathbb{C}/I$ is a category follows from $\mathbb{C}$ being a category.

2. We define $\mathrm{dom}\colon \mathbb{C}/I \to \mathbb{C}$ as follows: for a morphism h from $X \xrightarrow{f} I$ to $Y \xrightarrow{g} I$ in $\mathbb{C}/I$, we simply define $\mathrm{dom}(h) := h$. This immediately makes dom a functor from $\mathbb{C}/I$ to $\mathbb{C}$.

3. The claimed identification is obvious. Now fix a morphism h from $X \xrightarrow{f} I$ to $Y \xrightarrow{g} I$ in $\mathbf{Sets}/I$, so that the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

in **Sets** commutes. This requires that $g(h(x)) = f(x)$ for all $x \in X$. Identifying $X_i := f^{-1}(i)$ and $Y_i := g^{-1}(i)$ for all $i \in I$, we can identify h with a family of functions $(h_i)_{i\in I}$ such that $h_i(x) \in Y_i$ for all $x \in X_i$, for all $i \in I$.

$\square$

Exercise 1.4.4
Recall that for an arbitrary set A we write $A^\star$ for the set of finite sequencees $\langle a_0, \ldots, a_n\rangle$ of elements $a_i \in A$.

1. *Check that $A^\star$ carries a monoid structure given by concatenation of sequences, with the empty sequence $\langle\rangle$ as a neutral element.*

2. *Check that the assignment $A \mapsto A^\star$ yields a functor* **Sets** $\to$ **Mon** *by mapping a function $f\colon A \to B$ between sets to the function $f^\star\colon A^\star \to B^\star$ given by $\langle a_0, \ldots, a_n\rangle \mapsto \langle f(a_0), \ldots, f(a_n)\rangle$. (Be aware of what needs to be checked: $f^\star$ must be a monoid homomorphism, and $(-)^\star$ must preserve composition of functions and identity functions.)*

3. *Prove that $A^\star$ is the **free monoid on** A: there is the singleton-sequence insertion map $\eta\colon A \to A^\star$ which is universal among all mappings of A into a monoid. The latter means that for each monoid $(M, 0, +)$ and function $f\colon A \to M$ there is a unique monoid homomorphism $g\colon A^\star \to M$ with $g \circ \eta = f$.*

Solution.

1. Concatenation is associative because all the sequences under consideration are finite.

2. That $(-)^\star$ preserves composition and identity functions is obvious, so we just check that for a function $f\colon A \to B$, the map $f^\star\colon A^\star \to B\star$ is a monoid homomorphism. Fix finite sequences $\langle a_0, \ldots, a_n\rangle, \langle a'_0, \ldots, a'_k\rangle \in A^\star$. Then

$$\begin{aligned} f(\langle a_0, \ldots, a_n\rangle \cdot \langle a'_0, \ldots, a'_k\rangle) &= f(\langle a_0, \ldots, a_n, a'_0, \ldots, a'_k\rangle) \\ &= \langle f(a_0), \ldots, f(a_n), f(a'_0), \ldots, f(a'_k)\rangle \\ &= \langle f(a_0), \ldots, f(a_n)\rangle \cdot \langle f(a'_0), \ldots, f(a'_k)\rangle \end{aligned}$$

$$= f(\langle a_0, \dots, a_n \rangle) \cdot \langle a'_0, \dots a'_k \rangle)$$

and $f(\langle\rangle) = \langle\rangle$. So $f^\star$ is a monoid homomorphism.

3. Define $\eta\colon A \to A^\star$ by $\eta(a) \coloneqq \langle a \rangle$ for all $a \in A$. Fix a monoid $(M, 0, +)$ and a function $f\colon A \to M$. Define $g\colon A^\star \to M$ by

$$g(\langle\rangle) \coloneqq 0$$
$$g(\langle a_0, \dots, a_n \rangle) \coloneqq f(a_0) + \dots + f(a_n)$$

for all $\langle a_0, \dots, a_n \rangle \in A^\star$. This g is clearly a mononid homomorphism, using the associativity of $+$ in M. Observe that the diagram

$$\begin{tikzcd} & M \\ A \arrow[ur, "f"] \arrow[r, "\eta"'] & A^\star \arrow[u, "g"'] \end{tikzcd}$$

in **Sets** commutes: we have $f(a) = g(\eta(a))$ for all $a \in A$. Now suppose that there is another monoid homomorphism $h\colon A^\star \to M$ such that the diagram

$$\begin{tikzcd} & M \\ A \arrow[ur, "f"] \arrow[r, "\eta"'] & A^\star \arrow[u, "h"'] \end{tikzcd}$$

in **Sets** commutes. As $h\colon A^\star \to M$ is a monoid homomorphism and $f = h\eta$, we require that $h(\langle\rangle) = 0$ and

$$\begin{aligned} h(\langle a_0, \dots, a_n \rangle) &= h(\langle a_0 \rangle \cdot \ldots \cdot \langle a_n \rangle) \\ &= h(\langle a_0 \rangle) + \dots + h(\langle a_n \rangle) \\ &= h(\eta(a_0)) + \dots + h(\eta(a_n)) \\ &= f(a_0) + \dots + f(a_n) \\ &= g(\langle a_0, \dots, a_n \rangle), \end{aligned}$$

for all $\langle a_0, \dots, a_n \rangle \in A^\star$. Therefore $h = g$. □

Exercise 1.4.5
Recall from (1.3) the statements with exceptions of the form $S \to \{\bot\} \cup S \cup (S \times E)$.

1. *Prove that the assignment $X \mapsto \{\bot\} \cup X \cup (X \times E)$ is functorial, so that the statements are a coalgebra for this functor.*

2. *Show that all the operations $\mathsf{at}_1, \dots, \mathsf{at}_n, \mathsf{meth}_1, \dots, \mathsf{meth}_m$ of a class as in (1.10) can also be described as a single coalgebra, namely of the functor:*

$$X \mapsto D_1 \times \dots \times D_n \times \underbrace{(\{\bot\} \cup X \cup (X \times E)) \times \dots \times (\{\bot\} \cup X \cup (X \times E))}_{m \text{ times}}.$$

Solution.

1. Let $F\colon \mathbf{Sets} \to \mathbf{Sets}$ denote this assignment $F(X) \coloneqq \{\perp\} \cup X \cup (X \times E)$ where all unions are disjoint unions. We define F on morphisms as follows: for functions $f\colon X \to Y$, we define $F(f)\colon F(X) \to F(Y)$ to be the function

$$F(f)(x) \coloneqq \begin{cases} \perp, & \text{if } x = \perp, \\ f(x), & \text{if } x \in X, \\ (f(x'), e), & \text{if } x = (x', e) \text{ for some } (x', e) \in X \times E. \end{cases}$$

   Then $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(gf) = F(g)F(f)$ for all sets X and functions $X \xrightarrow{f} Y \xrightarrow{g} Z$.

2. The functor's definition on morphisms is similar in style with the previous part. □

Exercise 1.4.6
Recall the nexttime operator $\bigcirc$ for a sequence coalgebra $c\colon S \to \mathsf{Seq}(S) = \{\perp\} \cup (A \times S)$ from the previous section. Exercise 1.3.5.1 says that it forms a monotone function $\mathcal{P}(S) \to \mathcal{P}(S)$ — with respect to the inclusion order — and thus a functor. Check that invariants are precisely $\bigcirc$-coalgebras!

Solution. The $\bigcirc$-coalgebras are simply a subsets $U \subseteq S$ such that $U \subseteq \bigcirc U$. These are precisely what invariants are. □

2 Coalgebras of Polynomial Functors

2.1 Constructions on Sets

Exercise 2.1.1
Verify in detail the bijective correspondences (2.2), (2.6), (2.11) and (2.16).

Solution. Fix sets X, Y, Z. Following the notation of Equations (2.1), we associate a pair of functions $f\colon Z \to X$ and $g\colon Z \to Y$ to the function $\langle f, g\rangle\colon Z \to X \times Y$ given by $\langle f, g\rangle(z) := \langle f(z), g(z)\rangle$ for all $z \in Z$. Furthermore, we associate to any function $h\colon Z \to X \times Y$ a pair the functions $\pi_1 h\colon Z \to X$ and $\pi_2 h\colon Z \to Y$, where π_1 and π_2 are the relevant projections. Then $\langle \pi_1 h, \pi_2 h\rangle = h$ and $(\pi_1\langle f, g\rangle, \pi_2\langle f, g\rangle) = (f, g)$. This establishes the bijective correspondence (2.2).

Continue fixing sets X, Y, Z. Suppose, without loss of generality, that X and Y are disjoint, so that we may use $X \cup Y$ in place of $X + Y$. Following the notation of Equations (2.5), we associate a pair of functions $f\colon X \to Z$ and $g\colon Y \to Z$ to the function $[f, g]\colon X + Y \to Z$ given by

$$[f, g](w) := \begin{cases} f(w), & \text{if } w \in X, \\ g(w), & \text{if } w \in Y \end{cases}$$

for all $w \in X + Y$. Furthermore, to any function $h\colon X + Y \to Z$, we associate the pair of functions $h\kappa_1\colon X \to Z$ and $g\kappa_2\colon Y \to Z$, where κ_1 and κ_2 are the relevant coprojections. Then $[h\kappa_1, h\kappa_2] = h$ and $([f, g]\kappa_1, [f, g]\kappa_2) = (f, g)$. This establishes the bijective correspondence (2.6).

Continue fixing sets X, Y, Z. Following the notations of Equations (2.10), we associate a function $f\colon Z \times X \to Y$ to the function $\Lambda(f)\colon Z \to Y^X$ given by $\Lambda(f)(z) := f(z, -)$ for all $z \in Z$. Furthermore, to each function $g\colon Z \to Y^X$, we associate the function $U(g)\colon Z \times X \to Y$ given by $U(g)(z, x) := g(z)(x)$ for all $(z, x) \in Z \times X$. Then $\Lambda(U(g)) = g$ and $U(\Lambda(f)) = f$. So we have established the bijective correspondence (2.11).

Finally, fix sets X and Y. To each function $f\colon X \to \mathcal{P}(Y)$, we associate the relation

$$\mathsf{rel}(f) := \{\, (y, x) \in Y \times X \;:\; y \in f(x) \,\}.$$

Also, to each relation $R \subseteq Y \times X$, we associate the function $\mathsf{char}(R)\colon X \to \mathcal{P}(Y)$ given by

$$\mathsf{char}(R)(x) := \{\, y \in Y \;:\; R(y, x) \,\}$$

for all $y \in Y$. Then $\mathsf{rel}(\mathsf{char}(R)) = R$ and $\mathsf{char}(\mathsf{rel}(f)) = f$. We thus obtain the bijective correspondence (2.16). □

Exercise 2.1.2
Consider a poset $(D, \leq)$ as a category. Check that the product of two elements $d, e \in D$, if it exists, is the meet $d \wedge e$. And a coproduct of d, e, if it exists, is the join $d \vee e$.

Similarly, show that a final object is a top element $\top$ (with $d \leq \top$, for all $d \in D$) and that an initial object is a bottom element $\bot$ (with $\bot \leq d$, for all $d \in D$).

Solution. These follow immediately, as in a poset $(D, \leq)$, we have one (and only one) morphism $x \to y$ if and only if $x \leq y$, for $x, y \in D$, and that the only isomorphisms are identity morphisms. □

Exercise 2.1.3
Check that a product in a category $\mathbb{C}$ is the same as a coproduct in a category $\mathbb{C}^{\mathrm{op}}$.

Solution. Fix $X, Y, Z \in \mathsf{Obj}(\mathbb{C})$, and suppose the product $X \times Y$ exists in $\mathbb{C}$. For a pair of morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, we have the following diagram

$$\begin{tikzcd}
X & & Y \\
& X \times Y \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& Z \arrow[uul, bend left, "f"] \arrow[uur, bend right, "g"'] \arrow[u, dashed, "\exists! h"'] &
\end{tikzcd}$$

in $\mathbb{C}$ commuting. (When we assert that a diagram **commutes** without any further specification, we mean that every subdiagram in the present diagram commutes. In this case, we are asserting that there exists a unique morphism $Z \xrightarrow{h} X \times Y$ such that $\pi_1 h = f$ and $\pi_2 h = g$.) Thus we have the following diagram

$$\begin{tikzcd}
X \arrow[dr, "\pi_1"] \arrow[ddr, bend right, "f"'] & & Y \arrow[dl, "\pi_2"'] \arrow[ddl, bend left, "g"] \\
& X \times Y \arrow[d, dashed, "\exists! h"] & \\
& Z &
\end{tikzcd}$$

in $\mathbb{C}^{\mathrm{op}}$ commuting. This makes $X \times Y$ the coproduct of X and Y in $\mathbb{C}^{\mathrm{op}}$, with coprojections π_1 and π_2. Similarly, coproducts in $\mathbb{C}^{\mathrm{op}}$ correspond to products in $\mathbb{C}$. $\square$

Exercise 2.1.4
Fix a set A and prove that assignments $X \mapsto A \times X$, $X \mapsto A + X$ and $X \mapsto X^A$ are functorial and give rise to functors **Sets** $\to$ **Sets**.

Solution. Define $F, G, H\colon$ **Sets** $\to$ **Sets** as follows. For a set X,

$$\begin{aligned}
FX &\coloneqq A \times X, \\
GX &\coloneqq A + X, \text{ and} \\
HX &\coloneqq X^A.
\end{aligned}$$

For a function $f\colon X \to Y$, define the functions $Ff\colon A \times X \to A \times Y$, $Gf\colon A + X \to A + Y$, and $Hf\colon X^A \to Y^A$ as follows:

$$\begin{aligned}
(Ff)(a, x) &\coloneqq (a, f(x)), && \text{for all } (a, x) \in A \times X, \\
(Gf)(w) &\coloneqq \begin{cases} w, & \text{if } w \in A, \\ f(w), & \text{if } w \in X, \end{cases} && \text{for all } w \in A + X, \\
(Hf)(h) &\coloneqq fh, && \text{for all functions } h\colon A \to X,
\end{aligned}$$

where we have assumed, without loss of generality, that A and X are disjoint so that $X + A$ is treated as $X \cup A$.

Then, for any set X,

$$(F\mathrm{id}_X)(a, x) = (a, \mathrm{id}_X(x))$$

$$
\begin{aligned}
&= (a, x), && \text{for all } (a, x) \in A \times X, \\
(G\mathrm{id}_X)(w) &= \begin{cases} w, & \text{if } w \in A, \\ \mathrm{id}_X(w), & \text{if } w \in X, \end{cases} \\
&= w, && \text{for all } w \in A + X, \\
(H\mathrm{id}_X)(h) &= \mathrm{id}_X h \\
&= h, && \text{for all functions } h \colon A \to X,
\end{aligned}
$$

so $F\mathrm{id}_X = \mathrm{id}_{FX}$, $G\mathrm{id}_X = \mathrm{id}_{GX}$, and $H\mathrm{id}_X = \mathrm{id}_{HX}$. Now, for functions $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$
\begin{aligned}
(F(gf))(a, x) &= (a, g(f(x))) \\
&= (Fg)(a, f(x)) \\
&= (Fg \circ Ff)(a, x), && \text{for all } (a, x) \in A \times X, \\
(G(gf))(w) &= \begin{cases} w, & \text{if } w \in A, \\ g(f(w)), & \text{if } w \in X, \end{cases} \\
&= (Gg \circ Gf)(w), && \text{for all } w \in A + X, \\
(H(gf))(h) &= \lambda a \in A.(g(f(h(a)))), \\
&= (Hg)(fh) \\
&= (Hg \circ Hf)(h), && \text{for all functions } h \colon A \to X,
\end{aligned}
$$

so $F(gf) = (Fg)(Ff)$, $G(gf) = (Gg)(Gf)$, and $H(gf) = (Hg)(Hf)$. Thus F, G, and H are functors from **Sets** to **Sets**. $\square$

Exercise 2.1.5

Prove that the category **PoSets** *of partially ordered sets and monotone functions is a BiCCC. The definitions on the underlying sets X of a poset $(X, \leq)$ are like for ordinary sets but should be equipped with appropriate orders.*

Solution. The category **PoSets** has a terminal object, namely the singleton poset. Furthermore, given two posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$, we can define a partial ordering $\leq_{1\times 2}$ on the product $X_1 \times X_2$ by

$$(x_1, x_2) \leq_{1\times 2} (x_1', x_2') \quad \text{if and only if} \quad x_1 \leq x_1' \text{ and } x_2 \leq x_2'$$

for all $(x_1, x_2), (x_1', x_2') \in X_1 \times X_2$. This poset $(X_1 \times X_2, \leq_{1\times 2})$ has the universal property of the product: given another poset $(X_3, \leq_3)$ and a pair of monotone functions $f \colon (X_3, \leq_3) \to (X_1, \leq_1)$ and $g \colon (X_3, \leq_3) \to (X_2, \leq_2)$, we have the diagram

$$
\begin{tikzcd}
(X_1, \leq_1) & & (X_2, \leq_2) \\
& (X_1 \times X_2, \leq_{1\times 2}) \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& (X_3, \leq_3) \arrow[uul, bend left, "f"] \arrow[u, dashed, "\exists! h"'] \arrow[uur, bend right, "g"'] &
\end{tikzcd}
$$

in **PoSets** commuting, where π_1 and π_2 are the relevant projections (which are indeed monotone). The unique monotone function h is given by $h(x_3) \coloneqq (f(x_3), g(x_3))$ for all $x_3 \in X_3$. Therefore the category **PoSets** has finite products.

The category **PoSets** also has an initial object: the empty poset. Now, given two posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$, we can define a partial ordering $\leq_{1+2}$ on the coproduct $X_1 + X_2$ by

$$w \leq_{1+2} w' \quad \text{if and only if} \quad (w, w' \in X_1 \text{ and } w \leq_1 w') \text{ or } (w, w' \in X_2 \text{ and } w \leq_2 w')$$

for all $w, w' \in X_1 + X_2$, where we have assumed without loss of generality that X_1 and X_2 are disjoint so that $X_1 + X_2$ may be identified with $X_1 \cup X_2$. Then, given any other poset $(X_3, \leq_3)$ and a pair of monotone functions $f\colon (X_1, \leq_1) \to (X_3, \leq_3)$ and $g\colon (X_2, \leq_2) \to (X_3, \leq_3)$, we have the diagram

$$\begin{tikzcd}
 & (X_3, \leq_3) & \\
 & (X_1 + X_2, \leq_{1+2}) \arrow[u, dashed, "\exists! h"'] & \\
(X_1, \leq_1) \arrow[ur, "\kappa_1"'] \arrow[uur, bend left, "f"] & & (X_2, \leq_2) \arrow[ul, "\kappa_2"] \arrow[uul, bend right, "g"']
\end{tikzcd}$$

in **PoSets** commuting, where κ_1 and κ_2 are the relevant coprojections (which are also monotone). The unique monotone function h is given by

$$h(w) \coloneqq \begin{cases} f(w), & \text{if } w \in X_1, \\ g(w), & \text{if } w \in X_2, \end{cases}$$

for all $w \in X_1 + X_2$. Therefore **PoSets** also has finite coproducts.

Now we show that **PoSets** also has exponents. Fix any two posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$. We define a partial ordering $\leq_{2^1}$ on the set $X_2^{X_1}$ as follows:

$$f \leq_{2^1} g \quad \text{if and only if} \quad f(x) \leq_2 g(x) \text{ for all } x \in X_1.$$

for all functions $f, g\colon X_1 \to X_2$. Then, for any poset $(X_3, \leq_3)$ and monotone function $f\colon (X_3, \leq_3) \to (X_2, \leq_2)$, we have the diagram

$$\begin{tikzcd}
(X_2^{X_1}, \leq_{2^1}) & & (X_1, \leq_1) & \\
(X_3, \leq_3) \arrow[u, dashed, "\exists! g"] & & (X_1, \leq_1) \arrow[u, "\mathrm{id}_{(X_1, \leq_1)}"'] & \\
 & (X_2^{X_1} \times X_1, \leq_{2^1 \times 1}) \arrow[uul, "p_1"] \arrow[uur, "p_2"'] \arrow[rr, "\mathrm{ev}"] & & (X_2, \leq_2) \\
 & (X_3 \times X_1, \leq_{3 \times 1}) \arrow[uul, bend left, "\pi_1"] \arrow[u, dashed, "g \times \mathrm{id}_{(X_1, \leq_1)}"] \arrow[uur, bend right, "\pi_2"'] \arrow[urr, "f"'] & &
\end{tikzcd}$$

in **PoSets** commuting, where $\mathrm{ev}(h, x_1) \coloneqq h(x_1)$ for all $(h, x_1) \in X_2^{X_1} \times X_1$, and π_1, π_2, p_1, and p_2 are the relevant projections. The unique monotone function g is given by $g(x_3) \coloneqq \lambda x_1 \in X_1 . f(x_3, x_1)$. Therefore **PoSets** also has exponents. $\square$

Exercise 2.1.6
Consider the category **Mon** *of monoids with monoid homomorphisms between them.*

1. *Check that the singleton monoid* 1 *is both an initial and a final object in* **Mon***; this is called a zero object.*
2. *Given two monoids* $(M_1, +_1, 0_1)$ *and* $(M_2, +_2, 0_2)$*, one defines a product monoid* $M_1 \times M_2$ *with componentwise addition* $(x, y) + (x', y') = (x{+_1}, x', y +_2 y')$ *and unit* $(0_1, 0_2)$*. Prove that* $M_1 \times M_2$ *is again a monoid, which forms a product in the category* **Mon** *with the standard projection maps* $M_1 \xleftarrow{\pi_1} M_1 \times M_2 \xrightarrow{\pi_2} M_2$*.*
3. *Note that there are also coprojections* $M_1 \xrightarrow{\kappa_1} M_1 \times M_2 \xleftarrow{\kappa_2} M_2$*, given by* $\kappa_1(x) = (x, 0_2)$ *and* $\kappa_2(y) = (0_1, y)$*, which are monoid homomorphisms and which makes* $M_1 \times M_2$ *at the same time the coproduct of* M_1 *and* M_2 *in* **Mon** *(and hence a biproduct). Hint: Define the cotuple* $[f, g]$ *as* $x \mapsto f(x) + g(x)$*.*

Solution.

1. Any monoid homomorphism $f\colon (M_1, +_1, 0_1) \to (M_2, +_2, 0_2)$ must satisfy $f(0_1) = 0_2$, so the singleton monoid is initial in **Mon**. It is also the final in **Mon** because the constant map to the unit is a monoid homomorphism.
2. Fix $(m_1, m_2), (m_1', m_2'), (m_1'', m_2'') \in M_1 \times M_2$. Then, using the associativity of $+_1$ and $+_2$,

$$\begin{aligned}(m_1, m_2) + ((m_1', m_2') + (m_1'', m_2'')) &= (m_1, m_2) + (m_1' +_1 m_1'',\ m_2' +_2 m_2'') \\ &= (m_1 +_1 m_1' +_1 m_1'',\ m_2 +_2 m_2' +_2 m_2'') \\ &= (m_1 +_1 m_1',\ m_2 +_2 m_2') + (m_1'', m_2'').\end{aligned}$$

Furthermore,

$$\begin{aligned}(m_1, m_2) + (0_1, 0_2) &= (m_1 +_1 0_1, m_2{+_2}, 0_2) \\ &= (m_1, m_2)\end{aligned}$$

and, similarly, $(0_1, 0_2) + (m_1, m_2) = (m_1, m_2)$. So $(M_1 \times M_2, +, (0_1, 0_2))$ is a monoid.

We now show that $M_1 \times M_2$ really is the categorical product of M_1 and M_2 in **Mon**. Fix any other monoid $(M_3, +_3, 0_3)$ and a pair of monoid homomorphisms $f\colon M_3 \to M_1$ and $g\colon M_3 \to M_2$. We need the diagram

$$\begin{tikzcd}
M_1 & & M_2 \\
& M_1 \times M_2 \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& M_3 \arrow[uul, bend left, "f"] \arrow[uur, bend right, "g"'] \arrow[u, dashed, "\exists!\langle f,g\rangle"'] &
\end{tikzcd}$$

in **Mon** to commute. Indeed, we must have $\langle f, g\rangle(m_3) = (f(m_3), g(m_3))$ for all $m_3 \in M_3$. The fact that $\langle f, g\rangle\colon M_3 \to M_1 \times M_2$ is a monoid homomorphism follows from f and g being monoid homomorphisms.

3. Fix any monoid $(M_3, +_3, 0_3)$ and a pair of monoid homomorphisms $f\colon M_1 \to M_3$ and $g\colon M_2 \to M_3$. We need the diagram

$$\begin{tikzcd}
& M_3 & \\
& M_1 \times M_2 \arrow[u, dashed, "\exists![f,g]"'] & \\
M_1 \arrow[uur, bend left, "f"] \arrow[ur, "\kappa_1"'] & & M_2 \arrow[uul, bend right, "g"'] \arrow[ul, "\kappa_2"]
\end{tikzcd}$$

in **Mon** to commute. This time we define $[f,g]\colon M_1 \times M_2 \to M_3$ by

$$[f,g](m_1, m_2) \coloneqq f(m_1) +_3 g(m_2)$$

for all $(m_1, m_2) \in M_1 \times M_2$. That $[f,g]$ is a monoid homomorphism follows from f and g being monoid homomorphisms. Then

$$\begin{aligned}
([f,g] \circ \kappa_1)(m_1) &= [f,g](m_1, 0_1) \\
&= f(m_1) +_3 g(0_1) \\
&= f(m_1)
\end{aligned}$$

for all $m_1 \in M_1$. Similarly, $([f,g] \circ \kappa_2) = g$.

Now suppose there is another monoid homomorphism $h\colon M_1 \times M_2 \to M_3$ satisfying

$$h\kappa_1 = f \quad \text{and} \quad h\kappa_2 = g.$$

Then, for any $(m_1, m_2) \in M_1 \times M_2$,

$$\begin{aligned}
h(m_1, m_2) &= h(m_1, 0_2) +_3 h(0_1, m_2)) \\
&= h(\kappa_1(m_1)) +_3 h(\kappa_2(m_2)) \\
&= f(m_1) +_3 g(m_2) \\
&= [f,g](m_1, m_2).
\end{aligned}$$

Therefore $[f,g]$ is the unique monoid homomorphism making the diagram above commute. $\square$

Exercise 2.1.7
Show that in **Sets** *products distribute over coproducts, in the sense that the canonical maps*

$$(X \times Y) + (X \times Z) \xrightarrow{[\mathrm{id}_X \times \kappa_1, \mathrm{id}_X \times \kappa_2]} X \times (Y + Z)$$

$$0 \xrightarrow{\quad !\quad} X \times 0$$

are isomorphisms. Categories in which this is the case are called ***distributive****; see Cockett (1993) for more information on distributive categories in general and see Gumma, Hughes, and Schröder (2003) for an investigation of such distributivities in categories of coalgebras.*

Solution. In **Sets**, the initial object 0 is the empty set. Consequently, for any set X, the unique map $0 \xrightarrow{!} X \times 0$ is an isomorphism (in fact, ! is the identity morphism on 0) since $X \times 0 = 0$.

Now fix sets X, Y, and Z, and let $Y \xrightarrow{\kappa_1} Y+Z$ and $Z \xrightarrow{\kappa_2} Y+Z$ denote the appropriate coprojections. We may assume, without loss of generality, that Y and Z are disjoint, so that we may write $Y \cup Z$ in place of $Y + Z$, and have $\kappa_1 \colon Y \to Y \cup Z$ and $\kappa_2 \colon Z \to Y \cup Z$ be the appropriate inclusion functions.

The function $[\mathrm{id}_X \times \kappa_1, \mathrm{id}_X \times \kappa_2] \colon (X \times Y) + (X \times Z) \to X \times (Y + Z)$ is then given by

$$[\mathrm{id}_X \times \kappa_1, \mathrm{id}_X \times \kappa_2](x, w) = (x, w)$$

for all $(x, w) \in (X \times Y) + (X \times Z)$. This is clearly a bijection. □

Exercise 2.1.8

1. *Consider a category with finite products* $(\times, 1)$*. Prove that there are isomorphisms:*

$$X \times Y \cong Y \times X, \qquad (X \times Y) \times Z \cong X \times (Y \times Z), \qquad 1 \times X \cong X.$$

2. *Similarly, show that in a category with finite coproducts* $(+, 0)$ *one has*

$$X + Y \cong Y + X, \qquad (X + Y) + Z \cong X + (Y + Z), \qquad 0 + X \cong X.$$

(This means that both the finite product and coproduct structure in a category yield so-called symmetric monoidal structure. See Mac Lane (1978) or Borceux (1994) for more information.)

3. *Next, assume that our category also has exponents. Prove that*

$$X^0 \cong 1, \qquad X^1 \cong X, \qquad 1^X \cong 1.$$

And also that

$$Z^{X+Y} \cong Z^X \times Z^Y, \qquad Z^{X \times Y} \cong (Z^Y)^X, \qquad (X \times Y)^Z \cong X^Z \times Y^Z.$$

Solution.

1. Let $\mathbb{C}$ be a category with finite products. Fix $X, Y \in \mathsf{Obj}(\mathbb{C})$. Let $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ and $Y \xleftarrow{p_1} Y \times X \xrightarrow{p_2} X$ be the relevant projections. We have the diagram

$$\begin{tikzcd}
Y & & X \\
& Y \times X \arrow[ul, "p_1"'] \arrow[ur, "p_2"] & \\
X \arrow[uurr, "\mathrm{id}_X"] & & Y \arrow[uull, "\mathrm{id}_Y"'] \\
& X \times Y \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, dashed, "\langle \pi_2, \pi_1 \rangle"] &
\end{tikzcd}$$

in $\mathbb{C}$ commuting. We claim that the unique induced morphism $X \times Y \xrightarrow{\langle \pi_2, \pi_1 \rangle} X \times Y$ is an isomorphism. Of course, its inverse would be the similarly obtained morphism $Y \times X \xrightarrow{\langle p_2, p_1 \rangle} Y \times X$. Indeed, looking at the diagram

$$\begin{tikzcd}
X & & Y \\
& X \times Y \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
Y \arrow[uurr, bend right, "\mathrm{id}_Y"] & & X \arrow[uull, bend left, "\mathrm{id}_X"'] \\
& Y \times X \arrow[ul, "p_1"'] \arrow[ur, "p_2"] \arrow[uu, dashed, "\langle p_2, p_1 \rangle"] & \\
X \arrow[uurr, bend right, "\mathrm{id}_X"] & & Y \arrow[uull, bend left, "\mathrm{id}_Y"'] \\
& X \times Y \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] \arrow[uu, dashed, "\langle \pi_2, \pi_1 \rangle"] &
\end{tikzcd}$$

in $\mathbb{C}$, we see that

$$\begin{aligned}
\pi_1 \circ \langle p_2, p_1 \rangle \circ \langle \pi_2, \pi_1 \rangle &= p_2 \circ \langle \pi_2, \pi_1 \rangle \\
&= \pi_1
\end{aligned}$$

and, similarly, $\pi_2 \circ \langle p_2, p_1 \rangle \circ \langle \pi_2, \pi_1 \rangle = \pi_2$. Consequently, $\langle p_2, p_1 \rangle \circ \langle \pi_2, \pi_1 \rangle = \mathrm{id}_{X \times Y}$. Similarly, we obtain $\langle \pi_2, \pi_1 \rangle \circ \langle p_2, p_1 \rangle = \mathrm{id}_{Y \times X}$. Therefore we have an isomorphism $X \times Y \xrightarrow[\cong]{\langle \pi_2, \pi_1 \rangle} Y \times X$.

Now fix $X, Y, Z \in \mathsf{Obj}(\mathbb{C})$. Consider the products $X \times Y$ and $(X \times Y) \times Z$ as in the diagram

$$\begin{tikzcd}
X & & Y & Z \\
X \times Y \arrow[u, "p_1"] \arrow[urr, "p_2"] & & & \\
& (X \times Y) \times Z \arrow[ul, "\pi_1"] \arrow[uurr, "\pi_2"'] & &
\end{tikzcd}$$

in $\mathbb{C}$. These come with associated projections $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ and $X \times Y \xleftarrow{\pi_1} (X \times Y) \times Z \xrightarrow{\pi_2} Z$. We also have projections $X \xleftarrow{q_1} X \times (Y \times Z) \xrightarrow{q_2} Y \times Z$ and $Y \xleftarrow{r_1} Y \times Z \xrightarrow{r_2} Z$, as depicted in the diagram

$$
\begin{tikzcd}
X & Y & & Z \\
& & & Y \times Z \arrow[ull, "r_1"'] \arrow[u, "r_2"'] \\
& & X \times (Y \times Z) \arrow[uull, "q_1"] \arrow[ur, "q_2"']
\end{tikzcd}
$$

in $\mathbb{C}$. From these, we obtain the induced morphisms $(X \times Y) \times Z \xrightarrow{\langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle} X \times (Y \times Z)$ and $X \times (Y \times Z) \xrightarrow{\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle} (X \times Y) \times Z$.

$$
\begin{tikzcd}
X & & Y & & Z \\
X \times Y \arrow[u, "p_1"] \arrow[urr, "p_2"] & & & & Y \times Z \arrow[ull, "r_1"'] \arrow[u, "r_2"'] \\
& (X \times Y) \times Z \arrow[ul, "\pi_1"] \arrow[uuurrr, "\pi_2"] \arrow[rr, "\langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle"'] & & X \times (Y \times Z) \arrow[uuulll, "q_1"] \arrow[ur, "q_2"'] \arrow[ll, bend left, "\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle"]
\end{tikzcd}
$$

Then

$$
\begin{aligned}
p_1 \circ \pi_1 \circ \Big(\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle\Big) &= p_1 \circ \Big(\pi_1 \circ \langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle\Big) \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= p_1 \circ \langle q_1, r_1q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= q_1 \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= p_1\pi_1, \\
p_2 \circ \pi_1 \circ \Big(\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle\Big) &= p_2 \circ \Big(\pi_1 \circ \langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle\Big) \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= p_2 \circ \langle q_1, r_1q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= r_1 \circ q_2 \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= r_1 \circ \langle p_2\pi_1, \pi_2\rangle \\
&= p_2\pi_1, \text{ and} \\
\pi_2 \circ \Big(\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle\Big) &= \Big(\pi_2 \circ \langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle\Big) \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= r_2 \circ q_2 \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \\
&= r_2 \circ \langle p_2\pi_1, \pi_2\rangle \\
&= \pi_2.
\end{aligned}
$$

Thus $\langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle \circ \langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle = \mathrm{id}_{(X\times Y)\times Z}$. Via a similar calculation, we also obtain $\langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle \circ \langle\langle q_1, r_1q_2\rangle, r_2q_2\rangle = \mathrm{id}_{X\times(Y\times Z)}$. So we have an isomorphism $(X \times Y) \times Z \xrightarrow[\cong]{\langle p_1\pi_1, \langle p_2\pi_1, \pi_2\rangle\rangle} X \times (Y \times Z)$.

Now fix $X \in \mathsf{Obj}(\mathbb{C})$ and let 1 denote the terminal object in $\mathbb{C}$. We have the diagram

$$\begin{tikzcd}
1 & & X \\
& 1 \times X \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& X \arrow[u, dashed, "\langle f, \mathrm{id}_X \rangle"] \arrow[uul, dashed, bend left, "\exists! f"] \arrow[uur, bend right, "\mathrm{id}_X"'] &
\end{tikzcd}$$

in $\mathbb{C}$, where $1 \xleftarrow{\pi_1} 1 \times X \xrightarrow{\pi_2} X$ are the relevant projections, and $X \xrightarrow{f} 1$ is the unique morphism from X to 1. As the diagram commutes, we have $\pi_2 \circ \langle f, \mathrm{id}_X \rangle = \mathrm{id}_X$. Furthermore, $\pi_1 \circ (\langle f, \mathrm{id}_X \rangle \circ \pi_2) = \pi_1$ because 1 is the terminal object and we already have the morphism $1 \times X \xrightarrow{\pi_1} 1$. Moreover, $\pi_2 \circ (\langle f, \mathrm{id}_X \rangle \circ \pi_2) = \pi_2$.

$$\begin{tikzcd}
1 & & X \\
& 1 \times X \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& X \arrow[u, "\langle f, \mathrm{id}_X \rangle"] \arrow[uul, bend left, "f"] \arrow[uur, bend right, "\mathrm{id}_X"'] & \\
& 1 \times X \arrow[u, "\pi_2"] \arrow[uuul, bend left=60, "\pi_1"] \arrow[uuur, bend right=60, "\pi_2"'] &
\end{tikzcd}$$

Thus $\langle f, \mathrm{id}_X \rangle \circ \pi_2 = \mathrm{id}_{1 \times X}$. Therefore we have an isomorphism $1 \times X \xrightarrow[\cong]{\pi_2} X$.

2. This is dual to Exercise 2.1.8.1: coproducts in $\mathbb{C}$ coincide with products in $\mathbb{C}^{\mathrm{op}}$; the initial object in $\mathbb{C}$ is the terminal object in $\mathbb{C}^{\mathrm{op}}$; and isomorphisms in $\mathbb{C}$ are precisely isomorphisms in $\mathbb{C}^{\mathrm{op}}$.

3. Now suppose that the category $\mathbb{C}$ has all finite products, has all finite coproducts, and has exponents, i.e. $\mathbb{C}$ is a bicartesian closed category. Denote the initial and terminal objects of $\mathbb{C}$ by 0 and 1 respectively.

   Let us first show that $0 \times X \cong 0$ for all $X \in \mathsf{Obj}(\mathbb{C})$. Fix any $Y \in \mathsf{Obj}(\mathbb{C})$. For any morphism $0 \times X \xrightarrow{f} Y$, we have the following commuting diagram

$$\begin{tikzcd}
Y^X & & X & \\
0 \arrow[u, dashed, "\exists! \Lambda(f)"] & & X \arrow[u, "\mathrm{id}_X"'] & \\
& Y^X \times X \arrow[uul, "p_1"'] \arrow[uur, "p_2"] \arrow[rr, "\mathrm{ev}"] & & Y \\
& 0 \times X \arrow[u, dashed, "\Lambda(f) \times \mathrm{id}_X"] \arrow[uul, bend left, "\pi_1"] \arrow[uur, bend right, "\pi_2"'] \arrow[urr, "f"'] & &
\end{tikzcd}$$

in $\mathbb{C}$, where $0 \xleftarrow{\pi_1} 0 \times X \xrightarrow{\pi_2} X$ and $Y^X \xleftarrow{p_1} Y^X \times X \xrightarrow{p_2} X$ are the relevant projections and $Y^X \times X \xrightarrow{\text{ev}} Y$ is the appropriate evaluation morphism. Due to the initiality of 0, there is only one morphism $0 \to Y^X$ in $\mathbb{C}$. So there can be only one morphism $0 \times X \to Y$. Hence $0 \times X$ is also initial.

Now fix $X \in \mathsf{Obj}(\mathbb{C})$. Let us show that $X^0 \cong 1$. The diagram

$$\begin{tikzcd}
X^0 & & 0 & \\
1 \arrow[u, dashed, "\exists!\Lambda(\pi_2)"] & & 0 \arrow[u, "\mathrm{id}_0"'] & \\
& X^0 \times 0 \arrow[uul, "p_1"] \arrow[uur, "p_2"'] \arrow[rr, "\text{ev}"] & & 0 \\
& 1 \times 0 \arrow[uul, bend left, "\pi_1"] \arrow[uur, bend right, "\pi_2"'] \arrow[u, dashed, "\Lambda(\pi_2)\times\mathrm{id}_0"] \arrow[urr, "\pi_2"'] & &
\end{tikzcd}$$

in $\mathbb{C}$ commutes, where $1 \xleftarrow{\pi_1} 1 \times 0 \xrightarrow{\pi_2} 0$ and $X^0 \xleftarrow{p_1} X^0 \times 0 \xrightarrow{p_2} 0$ are the relevant projections and $X^0 \times 0 \xrightarrow{\text{ev}} 0$ is the relevant evaluation morphism. As 1 is the terminal object in $\mathbb{C}$, the composite morphism $1 \xrightarrow{\Lambda(\pi_2)} X^0 \xrightarrow{f} 1$ is equal to id_1, where $X^0 \xrightarrow{f} 1$ is the unique morphism from X^0 to 1. Also, the diagram

$$\begin{tikzcd}
X^0 & & 0 & \\
X^0 \arrow[u, "\Lambda(\pi_2)\circ f"] & & 0 \arrow[u, "\mathrm{id}_0"'] & \\
& X^0 \times 0 \arrow[uul, "p_1"] \arrow[uur, "p_2"'] \arrow[rr, "\text{ev}"] & & 0 \\
& X^0 \times 0 \arrow[uul, bend left, "\pi_1"] \arrow[uur, bend right, "\pi_2"'] \arrow[u, "(\Lambda(\pi_2)\circ f)\times\mathrm{id}_0"] \arrow[urr, "\text{ev}"'] & &
\end{tikzcd}$$

in $\mathbb{C}$ commutes because $X^0 \times 0 \cong 0$, as observed previously. Since we also have $\text{ev} \circ (\mathrm{id}_{X^0} \times \mathrm{id}_0) = \text{ev}$, the uniqueness clause in the universal property for exponential objects yields $\Lambda(\pi_2) \circ f = \mathrm{id}_{X^0}$. Therefore we have an isomorphism $1 \xrightarrow[\cong]{\Lambda(\pi_2)} X^0$.

Now fix $X \in \mathsf{Obj}(\mathbb{C})$. Let us show that $X^1 \cong X$. Let $X^1 \times 1 \xrightarrow{\text{ev}} X$ be the evaluation morphism obtained from the universal property of exponentials, and let $X \xleftarrow[\cong]{\pi_1} X \times 1 \xrightarrow{\pi_2} 1$ and $X^1 \xleftarrow[\cong]{p_1} X^1 \times 1 \xrightarrow{p_2} 1$ be the relevant projections, noting that π_1 and p_1 are both isomorphisms by our solution to Exercise 2.1.8.1. Then there exists a unique morphism $X \xrightarrow{\Lambda(\pi_1)} X^1$ such that

$\mathrm{ev}\circ(\Lambda(\pi_1)\times\mathrm{id}_1)=\pi_1$. That is, we have the diagram

$$\begin{tikzcd}
X^1 & & 1 & & \\
X \arrow[u, dashed, "\exists!\Lambda(\pi_1)"] & & 1 \arrow[u, "\mathrm{id}_1"'] & & \\
& X^1\times 1 \arrow[uul, "p_1", "\cong"'] \arrow[uur, "p_2"'] \arrow[rrr, "\mathrm{ev}"] & & & X \\
& X\times 1 \arrow[u, dashed, "\Lambda(\pi_1)\times\mathrm{id}_1"] \arrow[uul, bend left, "\pi_1", "\cong"'] \arrow[uur, bend right, "\pi_2"'] \arrow[urrr, "\pi_1"', "\cong"] & & &
\end{tikzcd}$$

in $\mathbb{C}$ commuting. Now, we claim that the diagram

$$\begin{tikzcd}
X^1\times 1 \arrow[r, "\mathrm{ev}"] & X \\
X\times 1 \arrow[u, "\Lambda(\pi_1)\times\mathrm{id}_1"] \arrow[ur, "\cong", "\pi_1"'] & \\
X^1\times 1 \arrow[u, "(\mathrm{ev}\circ p_1^{-1})\times\mathrm{id}_1"] \arrow[uur, "\mathrm{ev}"'] &
\end{tikzcd}$$

in $\mathbb{C}$ commutes. Indeed, the upper triangle commutes by definition of the morphisms $X^1\times 1\xrightarrow{\mathrm{ev}}X$ and $X\xrightarrow{\Lambda(\pi_1)}X^1$, and the lower triangle commutes because (the left square of) the diagram

$$\begin{tikzcd}
X & & 1 \\
& X\times 1 \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
X^1 \arrow[uu, "\mathrm{ev}\circ p_1^{-1}"] & & 1 \arrow[uu, "\mathrm{id}_1"'] \\
& X^1\times 1 \arrow[ul, "p_1"] \arrow[ur, "p_2"'] \arrow[uu, "(\mathrm{ev}\circ p_1^{-1})\times\mathrm{id}_1"] &
\end{tikzcd}$$

in $\mathbb{C}$ commutes by definition of the morphism $X^1\times 1\xrightarrow{(\mathrm{ev}\circ p_1^{-1})\times\mathrm{id}_1}X\times 1$. Hence $\Lambda(\pi_1)\circ\mathrm{ev}\circ p_1^{-1}=\mathrm{id}_{X^1}$, by the uniqueness clause in the universal property of exponentials, and thus the composite morphism $X^1\times 1\xrightarrow{(\mathrm{ev}\circ p_1^{-1}\times\mathrm{id}_1)}X\times 1\xrightarrow{\Lambda(\pi_1)\times\mathrm{id}_1}X^1\times 1$ equals $\mathrm{id}_{X^1\times 1}$. Also, the diagram

$$\begin{tikzcd}
X & X\times 1 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & 1 \\
& X^1\times 1 \arrow[ul, "\mathrm{ev}"] \arrow[u, "(\mathrm{ev}\circ p_1^{-1})\times\mathrm{id}_1"'] & \\
& X\times 1 \arrow[u, "\Lambda(\pi_1)\times\mathrm{id}_1"'] \arrow[uul, bend left, "\pi_1"] \arrow[uur, bend right, "\pi_2"'] &
\end{tikzcd}$$

in $\mathbb{C}$ commutes: the upper and lower left triangles commute as observed before; the right triangle commutes because 1 is the terminal object. So the composite morphism $X \times 1 \xrightarrow{\Lambda(\pi_1)\times \mathrm{id}_1} X^1 \times 1 \xrightarrow{(\mathrm{ev}\circ p_1^{-1})\times \mathrm{id}_1} X \times 1 = \mathrm{id}_{X\times 1}$. Consequently we have isomorphisms

$$X \underset{\cong}{\xleftarrow{\pi_1}} X \times 1 \underset{\cong}{\xrightarrow{\Lambda(\pi_1)\times \mathrm{id}_1}} X^1 \times 1 \underset{\cong}{\xrightarrow{p_1}} X^1,$$

yielding $X \cong X^1$.

Continue fixing $X \in \mathsf{Obj}(\mathbb{C})$. Let us now show that $1^X \cong 1$. Let $1^X \times X \xrightarrow{\mathrm{ev}} X$ be the relevant evaluation morphism, and let $1 \xleftarrow{\pi_1} 1 \times X \xrightarrow{\pi_2} X$ and $1^X \xleftarrow{p_1} 1^X \times X \xrightarrow{p_2} X$ be the relevant projections. Then we have the commuting diagram

$$\begin{tikzcd}
X^0 & & X & \\
1 \arrow[u, dashed, "\exists!\Lambda(\pi_2)"] & & X \arrow[u, "\mathrm{id}_X"'] & \\
& 1^X \times X \arrow[uul, "p_1"] \arrow[uur, "p_2"'] \arrow[rr, "\mathrm{ev}"] & & 1 \\
& 1 \times X \arrow[uul, bend left, "\pi_1"] \arrow[uur, bend right, "\pi_2"'] \arrow[urr, "\pi_2"'] \arrow[u, dashed, "\Lambda(\pi_2)\times \mathrm{id}_X"] & &
\end{tikzcd}$$

in $\mathbb{C}$. Now, letting $1^X \xrightarrow{f} 1$ be the unique morphism from 1^X to 1, we have that $f \circ \Lambda(\pi_2) = \mathrm{id}_1$ due to 1 being the terminal object. Furthermore, the diagram

$$\begin{tikzcd}
1^X \times X \arrow[r, "\mathrm{ev}"] & 1 \\
1^X \times X \arrow[u, "(\Lambda(\pi_2)\circ f)\times \mathrm{id}_X"] \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

in $\mathbb{C}$ also commutes because 1 is the terminal object. Thus we must have that $\Lambda(\pi_2) \circ f = \mathrm{id}_{1^X}$. Therefore we have the isomorphism $1^X \underset{\cong}{\xrightarrow{\Lambda(\pi_2)}} 1$.

From now onwards, we need to agree on some notation. For $A, B, C \in \mathsf{Obj}(\mathbb{C})$, we write $A^B \times B \xrightarrow{\mathrm{ev}_A^B} A$ for the evaluation morphism associated with the exponential object A^B. For a morphism $C \times B \xrightarrow{m} A$, we write $C \xrightarrow{\Lambda_A^B(m)} A^B$ for the unique morphism from C to A^B such that $\mathrm{ev}_A^B \circ (\Lambda_A^B(m) \times \mathrm{id}_B) = m$.

$$\begin{tikzcd}
A^B \times B \arrow[r, "\mathrm{ev}_A^B"] & A \\
C \times B \arrow[u, dashed, "\Lambda_A^B(m)\times \mathrm{id}_B"] \arrow[ur, "m"'] &
\end{tikzcd}$$

Furthermore, given a morphism $A \xrightarrow{m} B$ in $\mathbb{C}$, we define the morphism $A^C \xrightarrow{m^C} B^C$ to be the unique morphism from A^C to B^C satisfying $\mathrm{ev}_B^C \circ (m^C \times \mathrm{id}_C) = m \circ \mathrm{ev}_A^C$.

$$\begin{tikzcd}
B^C \times C \arrow[rr, "\mathrm{ev}_B^C"] & & B \\
& A \arrow[ur, "m"'] & \\
A^C \times C \arrow[uu, dashed, "m^C \times \mathrm{id}_C"] \arrow[ur, "\mathrm{ev}_A^C"'] & &
\end{tikzcd}$$

That is, $m^C \coloneqq \Lambda_B^C(m \circ \mathrm{ev}_A^C)$. Note that this makes the assignment $(-)^C \colon \mathbb{C} \to \mathbb{C}$ into a functor. Also, given morphisms $A \xleftarrow{g} A'$, $B \xrightarrow{h} B'$, and $A \times C \xrightarrow{m} B$ in $\mathbb{C}$, where $A, A', B, B', C \in \mathsf{Obj}(\mathbb{C})$, it is not difficult to see that $\Lambda_{B'}^C(h \circ m \circ (g \times \mathrm{id}_C)) = h^C \circ \Lambda_B^C(m) \circ g$ by looking at the commuting diagram

$$\begin{tikzcd}
(B')^C \times C \arrow[rr, "\mathrm{ev}_{B'}^C"] & & B' \\
B^C \times C \arrow[u, "h^C \times \mathrm{id}_C"] \arrow[r, "\mathrm{ev}_B^C"] & B \arrow[ur, "h"'] & \\
A \times C \arrow[u, "\Lambda_B^C(m) \times \mathrm{id}_C"] \arrow[ur, "m"'] & & \\
A' \times C \arrow[u, "g \times \mathrm{id}_C"] & &
\end{tikzcd}$$

in $\mathbb{C}$.

Let us take a detour and prove that the bicartesian closedness of $\mathbb{C}$ implies that products distribute over coproducts in $\mathbb{C}$ (from which Exercise 2.1.7 would also follow, since **Sets** is bicartesian closed). Fix $X, Y, Z \in \mathsf{Obj}(\mathbb{C})$. We already established that the unique map $0 \to 0 \times X$ is an isomorphism. We will now show that the canonical map $(Y \times X) + (Z \times X) \xrightarrow{[\kappa_1 \times \mathrm{id}_X,\ \kappa_2 \times \mathrm{id}_X]} (Y + Z) \times X$ is an isomorphism, where $Y \xrightarrow{\kappa_1} Y + Z \xleftarrow{\kappa_2} Z$ are the relevant coprojections. Further letting $Y \times X \xrightarrow{\iota_1} (Y \times X) + (Z \times X) \xleftarrow{\iota_2} Z \times X$ denote the relevant coprojections, we have the commuting diagrams

$$\begin{tikzcd}
(Y \times X) + (Z \times X) \arrow[rr, dashed, "\exists! [\kappa_1 \times \mathrm{id}_X,\ \kappa_2 \times \mathrm{id}_X]"] & & (Y + Z) \times X \\
Y \times X \arrow[u, "\iota_1"] \arrow[urr, "\kappa_1 \times \mathrm{id}_X"' near end] & & Z \times X \arrow[u, "\kappa_2 \times \mathrm{id}_X"'] \arrow[ull, "\iota_2" near end]
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y + Z \arrow[rr, dashed, "\exists! \left[\Lambda^X_{(Y \times X) + (Z \times X)}(\iota_1),\ \Lambda^X_{(Y \times X) + (Z \times X)}(\iota_2)\right]"] & & ((Y \times X) + (Z \times X))^X \\
Y \arrow[u, "\kappa_1"] \arrow[urr, "\Lambda^X_{(Y \times X) + (Z \times X)}(\iota_1)"' near end] & & Z \arrow[u, "\Lambda^X_{(Y \times X) + (Z \times X)}(\iota_2)"'] \arrow[ull, "\kappa_2" near end]
\end{tikzcd}$$

in $\mathbb{C}$, by the universal property of coproducts. Then, in the diagram

$$\begin{tikzcd}
& \big((Y\times X)+(Z\times X)\big)^X\times X \arrow[dl, "\mathrm{ev}^X_{(Y\times X)+(Z\times X)}"'] & \\
(Y\times X)+(Z\times X) \arrow[rr, "{[\kappa_1\times\mathrm{id}_X,\ \kappa_2\times\mathrm{id}_X]}"'] & & (Y+Z)\times X \arrow[ul, "{\big[\Lambda^X_{(Y\times X)+(Z\times X)}(\iota_1),\ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_2)\big]\times\mathrm{id}_X}"'] \\
Y\times X \arrow[u, "\iota_1"] \arrow[urr, bend right, "\kappa_1\times\mathrm{id}_X"'] & & Z\times X \arrow[u, "\kappa_2\times\mathrm{id}_X"'] \arrow[ull, bend left, "\iota_2"]
\end{tikzcd}$$

living in $\mathbb{C}$, we have the equalities

$$[\kappa_1\times\mathrm{id}_X,\ \kappa_2\times\mathrm{id}_X]\circ\iota_1=\kappa_1\times\mathrm{id}_X,$$
$$[\kappa_1\times\mathrm{id}_X,\ \kappa_2\times\mathrm{id}_X]\circ\iota_2=\kappa_2\times\mathrm{id}_X,$$

and

$$\mathrm{ev}^X_{(Y\times X)+(Z\times X)}\circ\big(\big[\Lambda^X_{(Y\times X)+(Z\times X)}(\iota_1),\ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_2)\big]\times\mathrm{id}_X\big)\circ(\kappa_1\times\mathrm{id}_X)=\iota_1,$$
$$\mathrm{ev}^X_{(Y\times X)+(Z\times X)}\circ\big(\big[\Lambda^X_{(Y\times X)+(Z\times X)}(\iota_1),\ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_2)\big]\times\mathrm{id}_X\big)\circ(\kappa_2\times\mathrm{id}_X)=\iota_2.$$

Consequently, by the universal property of coproducts,

$$\begin{aligned}&\mathrm{ev}^X_{(Y\times X)+(Z\times X)}\circ\big(\big[\Lambda^X_{(Y\times X)+(Z\times X)}(\iota_1),\ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_2)\big]\times\mathrm{id}_X\big)\circ[\kappa_1\times\mathrm{id}_X,\ \kappa_2\times\mathrm{id}_X]\\ &=\mathrm{id}_{(Y\times X)+(Z\times X)}.\end{aligned}$$

Now, let $f\coloneqq\mathrm{ev}^X_{(Y\times X)+(Z\times X)}\circ\big(\big[\Lambda^X_{(Y\times X)+(Z\times X)}(\iota_1),\ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_2)\big]\times\mathrm{id}_X\big)$,

$$\begin{tikzcd}
\big((Y\times X)+(Z\times X)\big)^X\times X \arrow[r, "\mathrm{ev}^X_{(Y\times X)+(Z\times X)}"] & (Y\times X)+(Z\times X) \\
(Y+Z)\times X \arrow[u, "{\big[\Lambda^X_{(Y\times X)+(Z\times X)}(\iota_1),\ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_2)\big]\times\mathrm{id}_X}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

so that $(Y+Z)\times X\xrightarrow{f}(Y\times X)+(Z\times X)$ is the unique morphism satisfying

$$\Lambda^X_{(Y\times X)+(Z\times X)}(f)=\big[\Lambda^X_{(Y\times X)+(Z\times X)}(\iota_1),\ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_2)\big].$$

We have already shown that $f\circ[\kappa_1\times\mathrm{id}_X,\ \kappa_2\times\mathrm{id}_X]=\mathrm{id}_{(Y\times X)+(Z\times X)}$. We will now show that $[\kappa_1\times\mathrm{id}_X,\ \kappa_2\times\mathrm{id}_X]\circ f=\mathrm{id}_{(Y+Z)\times X}$. This is equivalent to showing that

$$\mathrm{ev}^X_{(Y+Z)\times X}\circ\Big(\Lambda^X_{(Y+Z)\times X}\big([\kappa_1\times\mathrm{id}_X,\ \kappa_2\times\mathrm{id}_X]\circ f\big)\times\mathrm{id}_X\Big)=\mathrm{id}_{(Y+Z)\times X}.$$

$$
\begin{tikzcd}
\big((Y+Z)\times X\big)^X \times X \arrow[rr, "\mathrm{ev}^X_{(Y+Z)\times X}"] & & (Y+Z)\times X \\
& (Y\times X)+(Z\times X) \arrow[ur, "{[\kappa_1\times \mathrm{id}_X,\ \kappa_2\times \mathrm{id}_X]}"'] & \\
(Y+Z)\times X \arrow[uu, "\Lambda^X_{(Y+Z)\times X}\left([\kappa_1\times \mathrm{id}_X,\ \kappa_2\times \mathrm{id}_X]\circ f\right)\times \mathrm{id}_X"] \arrow[ur, "f"'] & &
\end{tikzcd}
$$

So let us proceed with showing the above equality.

$$
\begin{aligned}
&\mathrm{ev}^X_{(Y+Z)\times X} \circ \Big(\Lambda^X_{(Y+Z)\times X}\big([\kappa_1 \times \mathrm{id}_X,\ \kappa_2 \times \mathrm{id}_X] \circ f\big) \times \mathrm{id}_X\Big) \\
&= \mathrm{ev}^X_{(Y+Z)\times X} \circ \Big(\big([\kappa_1 \times \mathrm{id}_X,\ \kappa_2 \times \mathrm{id}_X]^X \circ \Lambda^X_{(Y\times X)+(Z\times X)}(f)\big) \times \mathrm{id}_X\Big) \\
&= \mathrm{ev}^X_{(Y+Z)\times X} \\
&\qquad \circ \bigg(\Big([\kappa_1 \times \mathrm{id}_X,\ \kappa_2 \times \mathrm{id}_X]^X \circ \big[\Lambda^X_{(Y\times X)+(Z\times X)}(\iota_1),\ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_2)\big]\Big) \times \mathrm{id}_X\bigg) \\
&= \mathrm{ev}^X_{(Y+Z)\times X} \circ \bigg(\Big[[\kappa_1 \times \mathrm{id}_X,\ \kappa_2 \times \mathrm{id}_X]^X \circ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_1), \\
&\qquad\qquad\qquad [\kappa_1 \times \mathrm{id}_X,\ \kappa_2 \times \mathrm{id}_X]^X \circ \Lambda^X_{(Y\times X)+(Z\times X)}(\iota_2)\Big] \times \mathrm{id}_X\bigg) \\
&= \mathrm{ev}^X_{(Y+Z)\times X} \circ \bigg(\Big[\Lambda^X_{(Y+Z)\times X}\big([\kappa_1 \times \mathrm{id}_X,\ \kappa_2 \times \mathrm{id}_X] \circ \iota_1\big), \\
&\qquad\qquad\qquad \Lambda^X_{(Y+Z)\times X}\big([\kappa_1 \times \mathrm{id}_X,\ \kappa_2 \times \mathrm{id}_X] \circ \iota_1\big)\Big] \times \mathrm{id}_X\bigg) \\
&= \mathrm{ev}^X_{(Y+Z)\times X} \circ \Big(\big[\Lambda^X_{(Y+Z)\times X}(\kappa_1 \times \mathrm{id}_X),\ \Lambda^X_{(Y+Z)\times X}(\kappa_2 \times \mathrm{id}_X)\big] \times \mathrm{id}_X\Big) \\
&= \mathrm{id}_{(Y+Z)\times X},
\end{aligned}
$$

where the last equality is due to the fact that

$$
\big[\Lambda^X_{(Y+Z)\times X}(\kappa_1 \times \mathrm{id}_X),\ \Lambda^X_{(Y+Z)\times X}(\kappa_2 \times \mathrm{id}_X)\big] = \Lambda^X_{(Y+Z)\times X}(\mathrm{id}_{(Y+Z)\times X}),
$$

which we shall now verify. Observe that

$$
\begin{aligned}
&\mathrm{ev}^X_{(Y+Z)\times X} \circ \big(\Lambda^X_{(Y+Z)\times X}(\kappa_1 \times \mathrm{id}_X) \times \mathrm{id}_X\big) \\
&= \kappa_1 \times \mathrm{id}_X \\
&= \mathrm{id}_{(Y+Z)\times X} \circ (\kappa_1 \times \mathrm{id}_X) \\
&= \mathrm{ev}^X_{(Y+Z)\times X} \circ \big(\Lambda^X_{(Y+Z)\times X}(\mathrm{id}_{(Y+Z)\times X}) \times \mathrm{id}_X\big) \circ (\kappa_1 \times \mathrm{id}_X) \\
&= \mathrm{ev}^X_{(Y+Z)\times X} \circ \Big(\big(\Lambda^X_{(Y+Z)\times X}(\mathrm{id}_{(Y+Z)\times X}) \circ \kappa_1\big) \times \mathrm{id}_X\Big).
\end{aligned}
$$

It follows that $\Lambda^X_{(Y+Z)\times X}(\kappa_1 \times \mathrm{id}_X) = \Lambda^X_{(Y+Z)\times X}(\mathrm{id}_{(Y+Z)\times X}) \circ \kappa_1$, by the universal property of exponents. Similarly, we have $\Lambda^X_{(Y+Z)\times X}(\kappa_2 \times \mathrm{id}_X) = \Lambda^X_{(Y+Z)\times X}(\mathrm{id}_{(Y+Z)\times X}) \circ \kappa_2$, and we thus obtain the desired fact.

Fix $X, Y, Z \in \mathsf{Obj}(\mathbb{C})$. Armed with the above observation that products distribute over coproducts, we are ready to show that $Z^{X+Y} \cong Z^X \times Z^Y$. Letting $X \xrightarrow{\kappa_1} X + Y \xleftarrow{\kappa_2} Y$ be the relevant coprojections, there exist unique morphisms $Z^X \xleftarrow{p_1} Z^{X+Y} \xrightarrow{p_1} Z^X$ making the diagrams

$$\begin{tikzcd}
Z^X \times X \arrow[rr, "\mathrm{ev}_Z^X"] & & Z \\
& Z^{X+Y} \times (X+Y) \arrow[ur, "\mathrm{ev}_Z^{X+Y}"'] & \\
Z^{X+Y} \times X \arrow[uu, "p_1 \times \mathrm{id}_X"] \arrow[ur, "\mathrm{id}_{Z^{X+Y}} \times \kappa_1"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
Z^X \times X \arrow[rr, "\mathrm{ev}_Z^X"] & & Z \\
& Z^{X+Y} \times (X+Y) \arrow[ur, "\mathrm{ev}_Z^{X+Y}"'] & \\
Z^{X+Y} \times X \arrow[uu, "p_2 \times \mathrm{id}_X"] \arrow[ur, "\mathrm{id}_{Z^{X+Y}} \times \kappa_2"'] & &
\end{tikzcd}$$

in $\mathbb{C}$ commute, namely $p_1 \coloneqq \Lambda_Z^X\big(\mathrm{ev}_Z^{X+Y} \circ (\mathrm{id}_{Z^{X+Y}} \times \kappa_1)\big)$ and $p_2 \coloneqq \Lambda_Z^X\big(\mathrm{ev}_Z^{X+Y} \circ (\mathrm{id}_{Z^{X+Y}} \times \kappa_2)\big)$. We will show that the object Z^{X+Y} along with the morphisms $Z^X \xleftarrow{p_1} Z^{X+Y} \xrightarrow{p_2} Z^Y$ serve as a categorical product of Z^X and Z^Y, which would yield $Z^{X+Y} \cong Z^X \times Z^Y$. Suppose we are given a pair of morphisms $Z^X \xleftarrow{f} A \xrightarrow{g} Z^Y$. We already know that there is an isomorphism $A \times (X+Y) \xrightarrow[\cong]{i} (A \times X) + (A \times Y)$ making the diagram

$$\begin{tikzcd}
& A \times (X+Y) \arrow[d, "i", "\cong"'] & \\
A \times X \arrow[ur, "\mathrm{id}_A \times \kappa_1"] \arrow[r, "\iota_1"'] & (A \times X) + (A \times Y) & A \times Y \arrow[ul, "\mathrm{id}_A \times \kappa_2"'] \arrow[l, "\iota_2"]
\end{tikzcd}$$

in $\mathbb{C}$ commute, where $A \times X \xrightarrow{\iota_1} (A \times X) + (A \times Y) \xleftarrow{\iota_2}$ are the relevant coprojections. By the universal property of exponentials, there exists a unique morphism $A \xrightarrow{h} Z^{X+Y}$ such that the diagram

$$\begin{tikzcd}
Z^X \times X \arrow[rr, "\mathrm{ev}_Z^X"] & & Z & Z^Y \times Y \arrow[l, "\mathrm{ev}_Z^Y"'] \\
& Z^{X+Y} \times (X+Y) \arrow[ur, "\mathrm{ev}_Z^{X+Y}"'] & & \\
& A \times (X+Y) \arrow[u, dashed, "h \times \mathrm{id}_{X+Y}"] \arrow[dr, "i", "\cong"'] & & \\
A \times X \arrow[uuu, "f \times \mathrm{id}_X"] \arrow[ur, "\mathrm{id}_A \times \kappa_1"'] \arrow[rr, "\iota_1"'] & & (A \times X) + (A \times Y) \arrow[uuu, "{[\mathrm{ev}_Z^X \circ (f \times \mathrm{id}_X),\ \mathrm{ev}_Z^Y \circ (g \times \mathrm{id}_Y)]}"] & A \times Y \arrow[uuu, "g \times \mathrm{id}_Y"'] \arrow[l, "\iota_2"] \arrow[ull, "\mathrm{id}_A \times \kappa_2"']
\end{tikzcd}$$

in $\mathbb{C}$ commutes, namely $h \coloneqq \Lambda_Z^{X+Y}\big([\mathrm{ev}_Z^X \circ (f \times \mathrm{id}_X),\ \mathrm{ev}_Z^Y \circ (g \times \mathrm{id}_Y)] \circ i\big)$. Hence

$$\mathrm{ev}_Z^X \circ (f \times \mathrm{id}_X) = \mathrm{ev}_Z^{X+Y} \circ (h \times \mathrm{id}_{X+Y}) \circ (\mathrm{id}_A \times \kappa_1)$$

$$
\begin{aligned}
&= \mathrm{ev}_Z^{X+Y} \circ (h \times \kappa_1) \\
&= \mathrm{ev}_Z^{X+Y} \circ (\mathrm{id}_{Z^{X+Y}} \times \kappa_1) \circ (h \times \mathrm{id}_X) \\
&= \mathrm{ev}_Z^{X} \circ (p_1 \times \mathrm{id}_X) \circ (h \times \mathrm{id}_X) \\
&= \mathrm{ev}_Z^{X} \circ (p_1 h \times \mathrm{id}_X),
\end{aligned}
$$

and so $f = p_1 h$ by the universal property of exponents. Similarly, $g = p_2 h$. Now, for any morphism $A \xrightarrow{k} Z^{X+Y}$ in $\mathbb{C}$ satisfying $f = p_1 k$ and $g = p_2 k$, then we get the equalities

$$\mathrm{ev}_Z^X \circ (f \times \mathrm{id}_X) = \mathrm{ev}_Z^{X+Y} \circ (k \times \mathrm{id}_{X+Y}) \circ (\mathrm{id}_A \times \kappa_1)$$

and

$$\mathrm{ev}_Z^Y \circ (g \times \mathrm{id}_X) = \mathrm{ev}_Z^{X+Y} \circ (k \times \mathrm{id}_{X+Y}) \circ (\mathrm{id}_A \times \kappa_2).$$

From these, it follows that

$$\mathrm{ev}_Z^{X+Y} \circ (h \times \mathrm{id}_{X+Y}) \circ i^{-1} \circ \iota_1 = \mathrm{ev}_Z^{X+Y} \circ (k \times \mathrm{id}_{X+Y}) \circ i^{-1} \circ \iota_1$$

and

$$\mathrm{ev}_Z^{X+Y} \circ (h \times \mathrm{id}_{X+Y}) \circ i^{-1} \circ \iota_2 = \mathrm{ev}_Z^{X+Y} \circ (k \times \mathrm{id}_{X+Y}) \circ i^{-1} \circ \iota_2.$$

From the universal property of coproducts and the fact that i^{-1} is an isomorphism, the above two equalities let us obtain $\mathrm{ev}_Z^{X+Y} \circ (h \times \mathrm{id}_{X+Y}) = \mathrm{ev}_Z^{X+Y} \circ (k \times \mathrm{id}_{X+Y})$. The universal property of exponents then implies that $h = k$. Therefore $Z^X \xleftarrow{p_1} Z^{X+Y} \xrightarrow{p_2} Z^Y$ serves as a categorical product of Z^X and Z^Y, giving $Z^{X+Y} \cong Z^X \times Z^Y$.

Let us move on to showing that $Z^{X \times Y} \cong (Z^Y)^X$. From our solution to Exercise 2.1.8.1, we know that there are isomorphisms $(A \times X) \times Y \underset{\cong}{\xrightarrow{i}} A \times (X \times Y)$ and $((Z^Y)^X \times X) \times Y \underset{\cong}{\xrightarrow{j}} (Z^Y)^X \times (X \times Y)$ such that for any morphism $A \xrightarrow{k} (Z^Y)^X$ the diagram

$$
\begin{tikzcd}
(A \times X) \times Y \arrow[r, "(k \times \mathrm{id}_X) \times \mathrm{id}_Y"] & ((Z^Y)^X \times X) \times Y \arrow[d, "j^{-1}", "\cong"'] \\
A \times (X \times Y) \arrow[u, "i^{-1}", "\cong"'] \arrow[r, "k \times \mathrm{id}_{X \times Y}"'] & (Z^Y)^X \times (X \times Y)
\end{tikzcd}
$$

in $\mathbb{C}$ commutes, i.e. $j^{-1} \circ ((k \times \mathrm{id}_X) \times \mathrm{id}_Y) \circ i^{-1} = k \times \mathrm{id}_{X \times Y}$. Now suppose we are given a morphism $A \times (X \times Y) \xrightarrow{f} Z$. Then the diagram

$$
\begin{tikzcd}
(Z^Y)^X \times (X \times Y) \arrow[r, "j", "\cong"'] & ((Z^Y)^X \times X) \times Y \arrow[r, "\mathrm{ev}_{Z^Y}^X \times \mathrm{id}_Y"] & Z^Y \times Y \arrow[r, "\mathrm{ev}_Z^Y"] & Z \\
 & (A \times X) \times Y \arrow[u, "\left(\Lambda_{Z^Y}^X\left(\Lambda_Z^Y(fi)\right) \times \mathrm{id}_X\right) \times \mathrm{id}_Y"] \arrow[ur, "\Lambda_Z^Y(fi) \times \mathrm{id}_Y"'] \arrow[r, "\cong", "i"'] & A \times (X \times Y) \arrow[ur, "f"'] &
\end{tikzcd}
$$

in $\mathbb{C}$ commutes. This yields a unique morphism $A \xrightarrow{h} (Z^Y)^X$ satisfying

$$\left(\mathrm{ev}_Z^Y \circ \left(\mathrm{ev}_{Z^Y}^X \times \mathrm{id}_Y\right) \circ j\right) \circ \left(h \times \mathrm{id}_{X \times Y}\right) = f,$$

namely $h = \Lambda^X_{Z^Y}\big(\Lambda^Y_Z(fi)\big)$. So the object $(Z^X)^Y$ with the morphism $(Z^Y)^X \times (X \times Y) \xrightarrow{\mathrm{ev}^Y_Z \circ (\mathrm{ev}^X_{Z^Y} \times \mathrm{id}_Y) \circ j}$ Z serve as the exponential object $Z^{X\times Y}$ and its evaluation morphism. Hence $(Z^Y)^X \cong Z^{X\times Y}$.

Finally, let us show that $(X\times Y)^Z \cong X^Z \times Y^Z$. Let $X \xleftarrow{\pi_1} X\times Y \xrightarrow{\pi_2} Y$ be the relevant projections. Suppose we are given a morphism $A \times Z \xrightarrow{f} X \times Y$. Then we obtain morphisms $X \xleftarrow{\pi_1 f} A \times Z \xrightarrow{\pi_2 f} Y$, from which we obtain the two unique morphisms $X^Z \xleftarrow{\Lambda^Z_X(\pi_1 f)} A \xrightarrow{\Lambda^Z_Y(\pi_2 f)} Y^Z$ satisfying

$$\mathrm{ev}^Z_X \circ \Lambda^Z_X(\pi_1 f) = \pi_1 f \quad \text{and} \quad \mathrm{ev}^Z_Y \circ \Lambda^Z_Y(\pi_2 f) = \pi_2 f.$$

Letting $X^Z \xleftarrow{p_1} X^Z \times Y^Z \xrightarrow{p_2} Y^Z$ be the relevant projections, an elementary calculation shows that the diagram

$$\begin{tikzcd}
(X^Z \times Y^Z) \times Z \arrow[rr, "\left\langle \mathrm{ev}^Z_X \circ (p_1 \times \mathrm{id}_Z),\ \mathrm{ev}^Z_Y \circ (p_2 \times \mathrm{id}_Z) \right\rangle"] & & X \times Y \\
A \times Z \arrow[u, "\left\langle \Lambda^Z_X(\pi_1 f),\ \Lambda^Z_Y(\pi_2 f) \right\rangle \times \mathrm{id}_Z"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

in $\mathbb{C}$ commutes. An arbitrary morphism $A \xrightarrow{h} X^Z \times Y^Z$ satisfying

$$\left\langle \mathrm{ev}^Z_X \circ (p_1 \times \mathrm{id}_Z),\ \mathrm{ev}^Z_Y \circ (p_2 \times \mathrm{id}_Z) \right\rangle \circ (h \times \mathrm{id}_Z) = f = \langle \pi_1 f, \pi_2 f \rangle$$

must then satisfy $p_1 h = \Lambda^Z_X(\pi_1 f)$ and $p_2 h = \Lambda^Z_X(\pi_2 f)$. This yields $h = \left\langle \Lambda^Z_X(\pi_1 f),\ \Lambda^Z_Y(\pi_2 f) \right\rangle$. So the object $X^Z \times Y^Z$ together with the morphism $(X^Z \times Y^Z) \times Z \xrightarrow{\left\langle \mathrm{ev}^Z_X \circ (p_1 \times \mathrm{id}_Z),\ \mathrm{ev}^Z_Y \circ (p_2 \times \mathrm{id}_Z) \right\rangle} X \times Y$ serve as the exponential object $(X \times Y)^Z$ and its evaluation morphism. Therefore $(X \times Y)^Z \cong X^Z \times Y^Z$. □

Exercise 2.1.9
Show that the finite powerset also forms a functor $\mathcal{P}_{\mathrm{fin}} \colon \mathbf{Sets} \to \mathbf{Sets}$.

Solution. The proof that $\mathcal{P}_{\mathrm{fin}} \colon \mathbf{Sets} \to \mathbf{Sets}$ is a functor is identical to the proof that the usual power set operation $\mathcal{P} \colon \mathbf{Sets} \to \mathbf{Sets}$ is a functor. Given a function $f \colon X \to Y$, the function $\mathcal{P}_{\mathrm{fin}} f \colon \mathcal{P}_{\mathrm{fin}} X \to \mathcal{P}_{\mathrm{fin}} Y$ sends finite subsets $A \subseteq X$ to their image under f. That is, for finite subsets $A \subseteq X$, we define

$$(\mathcal{P}_{\mathrm{fin}} f)(A) \coloneqq \{\, f(x) \ : \ x \in A \,\},$$

which is indeed a finite set.

It is clear that $\mathcal{P}_{\mathrm{fin}} \mathrm{id}_X = \mathrm{id}_{\mathcal{P}_{\mathrm{fin}} X}$ for all sets X. Now given functions $f \colon X \to Y$ and $g \colon Y \to Z$,

$$\begin{aligned}
(\mathcal{P}_{\mathrm{fin}}(gf))(A) &= \{\, g(f(x)) \ : \ x \in A \,\} \\
&= \{\, g(y) \ : \ y \in (\mathcal{P}_{\mathrm{fin}} f)(A) \,\} \\
&= (\mathcal{P}_{\mathrm{fin}} g)((\mathcal{P}_{\mathrm{fin}} f)(A)) \\
&= (\mathcal{P}_{\mathrm{fin}} g \circ \mathcal{P}_{\mathrm{fin}} f)(A)
\end{aligned}$$

for all finite subsets $A \subseteq X$. Thus $\mathcal{P}_{\mathrm{fin}}(gf) = (\mathcal{P}_{\mathrm{fin}} g)(\mathcal{P}_{\mathrm{fin}} f)$. □

Exercise 2.1.10
Check that

$$\mathcal{P}(0) \cong 1, \qquad \mathcal{P}(X+Y) \cong \mathcal{P}(X) \times \mathcal{P}(Y).$$

And similarly for the finite powerset $\mathcal{P}_{\mathrm{fin}}$ instead of $\mathcal{P}$. This property says that $\mathcal{P}$ and $\mathcal{P}_{\mathrm{fin}}$ are 'additive'; see Coumans and Jacobs (2013).

Solution. Let 0 and 1 respectively denote the initial and terminal objects in **Sets**. Then $\mathcal{P}(0) = \mathcal{P}_{\text{fin}}(0) = \mathcal{P}(\varnothing) = \{\varnothing\} \cong 1$.

Now fix sets X and Y and suppose, without loss of generality, that X and Y are disjoint so that we can write $X + Y = X \cup Y$. Then, we have a bijection $f\colon \mathcal{P}(X+Y) \to \mathcal{P}X \times \mathcal{P}Y$ defined by

$$f(A) := \big(\{\, z \in A \ :\ z \in X \,\},\ \{\, z \in A \ :\ z \in Y \,\}\big)$$

for all $A \subseteq X + Y$. This is indeed a bijection as it has inverse $f^{-1}\colon \mathcal{P}X \times \mathcal{P}Y \to \mathcal{P}(X+Y)$ defined by

$$f^{-1}(A, B) := A \cup B.$$

The proof that $\mathcal{P}_{\text{fin}}(X+Y) \cong \mathcal{P}_{\text{fin}}X \times \mathcal{P}_{\text{fin}}Y$ is similar. □

Exercise 2.1.11
Notice that a power set $\mathcal{P}(X)$ can also be understood as exponent 2^X, where $2 = \{0, 1\}$. Check that the exponent functoriality gives rise to the contravariant powerset $\mathbf{Sets}^{\text{op}} \to \mathbf{Sets}$.

Solution. The identification of $\mathcal{P}(X)$ with 2^X is via the isomorphism $\alpha_X\colon \mathcal{P}(X) \to 2^X$ defined by

$$\alpha_X(A) := \lambda x \in X. \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

for all $A \subseteq X$.

Fix a function $f\colon X \to Y$. The function $2^f\colon 2^Y \to 2^X$ is given by

$$(2^f)(k) := \lambda x \in X. k(f(x)),$$

for all functions $k\colon Y \to 2$. We then see that $\alpha_X^{-1} \circ 2^f \circ \alpha_Y\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ satisfies

$$\begin{aligned}
\big(\alpha_X^{-1} \circ 2^f \circ \alpha_Y\big)(B) &= \big(\alpha_X^{-1} \circ 2^f\big)\left(\lambda y \in Y. \begin{cases} 1, & \text{if } y \in B, \\ 0, & \text{if } y \notin B \end{cases}\right) \\
&= \alpha_X^{-1}\left(\lambda x \in X. \begin{cases} 1, & \text{if } f(x) \in B, \\ 0, & \text{if } f(x) \notin B \end{cases}\right) \\
&= \{\, x \in X \ :\ f(x) \in B \,\}
\end{aligned}$$

for all $B \subseteq Y$. This is precisely how the contravariant power set functor is defined on morphisms. □

Exercise 2.1.12
Consider a function $f\colon X \to Y$. Prove that

1. *The direct image $\mathcal{P}(f) = \coprod_f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ preserves all joins and that the inverse image $f^{-1}(-)\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves not only joins but also meets and negation (i.e. all the Boolean structure).*

2. *There is a Galois connection $\coprod_f(U) \subseteq V \iff U \subseteq f^{-1}(V)$, as claimed in (2.15).*

3. *There is a product function $\prod_f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ given by $\prod_f(U) = \{\, y \in Y \mid \forall x \in X. f(x) = y \Rightarrow x \in U \,\}$, with a Galois connection $f^{-1}(V) \subseteq U \iff V \subseteq \prod_f(U)$.*

Solution.

1. For a collection $\{A_i\}_{i\in I}$ of subsets of X, we see that

$$\begin{aligned}(\mathcal{P}f)\left(\bigcup_{i\in I} A_i\right) &= \left\{\, f(x) \;:\; x \in \bigcup_{i\in I} A_i \,\right\} \\ &= \bigcup_{i\in I}\{\, f(x) \;:\; x \in A_i \,\} \\ &= \bigcup_{i\in I}(\mathcal{P}f)(A_i).\end{aligned}$$

So $\mathcal{P}f$ preserves all joins. Furthermore, for a collection $\{B_j\}_{j\in J}$ of subsets of X,

$$\begin{aligned}f^{-1}\left(\bigcup_{j\in J} B_j\right) &= \left\{\, x \in X \;:\; f(x) \in \bigcup_{j\in J} B_j \,\right\} \\ &= \bigcup_{j\in J}\{\, x \in X \;:\; f(x) \in B_j \,\} \\ &= \bigcup_{j\in J} f^{-1}(B_j).\end{aligned}$$

So $f^{-1}(-)$ also preserves all joins. Moreover,

$$\begin{aligned}f^{-1}\left(\bigcap_{j\in J} B_j\right) &= \left\{\, x \in X \;:\; f(x) \in \bigcap_{j\in J} B_j \,\right\} \\ &= \bigcap_{j\in J}\{\, x \in X \;:\; f(x) \in B_j \,\} \\ &= \bigcap_{j\in J} f^{-1}(B_j).\end{aligned}$$

So $f^{-1}(-)$ preserves all meets. Also, for any subset $B \subseteq Y$,

$$\begin{aligned}f^{-1}(Y \setminus B) &= \{\, x \in X \;:\; f(x) \in Y \setminus B \,\} \\ &= X \setminus \{\, x \in X \;:\; f(x) \in B \,\} \\ &= X \setminus f^{-1}(B).\end{aligned}$$

So $f^{-1}(-)$ preserves all negations.

2. Fix a pair of subsets $U \subseteq X$ and $V \subseteq Y$. Then

$$\begin{aligned}(\mathcal{P}f)(U) \subseteq V \quad &\text{if and only if} \quad \{\, f(x) : x \in U \,\} \subseteq V \\ &\text{if and only if} \quad \text{for all } x \in U \text{ we have } f(x) \in V \\ &\text{if and only if} \quad U \subseteq \{\, x \in X : f(x) \in V \,\} \\ &\text{if and only if} \quad U \subseteq f^{-1}(V),\end{aligned}$$

as claimed.

3. Fix a pair of subsets $U \subseteq X$ and $V \subseteq Y$. Then

$$f^{-1}(V) \subseteq U \quad \text{if and only if} \quad \{\, x \in X : f(x) \in V \,\} \subseteq U$$

$$
\begin{aligned}
&\text{if and only if} \quad \text{for all } x \in X \text{ with } f(x) \in V \text{ we have } x \in U \\
&\text{if and only if} \quad V \subseteq \{\, y \in Y : \text{for all } x \in X \text{ with } f(x) = y \text{ we have } x \in U \,\} \\
&\text{if and only if} \quad V \subseteq \prod_f (U),
\end{aligned}
$$

as desired. □

Exercise 2.1.13
Assume a category $\mathbb{C}$ has arbitrary, set-indexed coproducts $\coprod_{i \in I} X_i$. Demonstrate, as in the proof of Proposition 2.1.5, that the category $\mathbf{CoAlg}(F)$ of coalgebras of a functor $F\colon \mathbb{C} \to \mathbb{C}$ then also has such coproducts.

Solution. Let I be a non-empty set and fix an I-indexed tuple $(X_i \xrightarrow{\alpha_i} FX_i)_{i \in I}$ of F-coalgebras. Let $C \coloneqq \coprod_{i \in I} X_i$ be the coproduct of $(X_i)_{i \in I}$ in $\mathbb{C}$ and, for $i \in I$, let $X_i \xrightarrow{\kappa_i} C$ denote the appropriate coprojection. We have the collection of morphisms $(X_i \xrightarrow{(F\kappa_i)\alpha_i} FC)_{i \in I}$. So there exists a unique morphism $\gamma\colon C \to FC$ such that $\gamma\kappa_i = (F\kappa_i)\alpha_i$ for all $i \in I$. That is, the diagram

$$
\begin{tikzcd}
FC & & & & & \\
& FX_0 \arrow[ul, "F\kappa_0"'] & FX_1 \arrow[ull, "F\kappa_1"'] & \cdots & FX_i \arrow[ullll, bend right=10, "F\kappa_i"'] & \cdots \\
C \arrow[uu, dashed, "\exists!\gamma"] & & & & & \\
& X_0 \arrow[ul, "\kappa_0"] \arrow[uu, "\alpha_0"] & X_1 \arrow[ull, "\kappa_1"] \arrow[uu, "\alpha_1"] & \cdots & X_i \arrow[ullll, bend right=10, "\kappa_i"] \arrow[uu, "\alpha_i"] & \cdots
\end{tikzcd}
$$

in $\mathbb{C}$ commutes. Consequently, we have a collection of homomorphisms of F-coalgebras $((X_i, \alpha_i) \xrightarrow{\kappa_i} (C, \gamma))_{i \in I}$.

For the universal property, we argue as in the following diagram.

$$
\begin{tikzcd}
& FC \arrow[dl, dashed, "Fg"'] & & & & & \\
FY & & FX_0 \arrow[ul, "F\kappa_0"'] \arrow[ll, bend right, "Ff_0"'] & FX_1 \arrow[ull, "F\kappa_1"'] \arrow[lll, bend left, "Ff_1"'] & \cdots & FX_i \arrow[ullll, bend right=10, "F\kappa_i"'] \arrow[lllll, bend left, "Ff_i"'] & \cdots \\
& C \arrow[uu, "\gamma"] \arrow[dl, dashed, "\exists!g"'] & & & & & \\
Y \arrow[uu, "\beta"] & & X_0 \arrow[ul, "\kappa_0"] \arrow[uu, "\alpha_0"] \arrow[ll, bend right, "f_0"'] & X_1 \arrow[ull, "\kappa_1"] \arrow[uu, "\alpha_1"] \arrow[lll, bend left, "f_1"'] & \cdots & X_i \arrow[ullll, bend right=10, "\kappa_i"] \arrow[uu, "\alpha_i"] \arrow[lllll, bend left, "f_i"] & \cdots
\end{tikzcd}
$$

Suppose we are given another F-coalgebra $Y \xrightarrow{\beta} FY$ and a collection of homomorphisms of F-coalgebras $((X_i, \alpha_i) \xrightarrow{f_i} (Y, \beta))_{i \in I}$. Then, as C is the coproduct of $(X_i)_{i \in I}$ in $\mathbb{C}$, there is a unique morphism $C \xrightarrow{g} Y$ in $\mathbb{C}$ such that $g\kappa_i = f_i$ for all $i \in I$.

We now need to verify that g is actually a homomorphism of F-coalgebras from (C, γ) to (Y, β). We will use the universal property of C as the coproduct in $\mathbb{C}$: for all $i \in I$, we have

$$\begin{aligned}
\beta g \kappa_i &= \beta f_i, && \text{since } g\kappa_i = f_i, \\
&= (Ff_i)\alpha_i, && \text{since } f_i \text{ is a homomorphism from } (X_i, \alpha_i) \text{ to } (Y, \beta), \\
&= (Fg)(F\kappa_i)\alpha_i, && \text{from } g\kappa_i = f_i \text{ and the functoriality of } F, \\
&= (Fg)\gamma\kappa_i, && \text{since } \kappa_i \text{ is a homomorphism from } (X_i, \alpha_i) \text{ to } (C, \gamma).
\end{aligned}$$

Therefore $\beta g = (Fg)\gamma$, i.e. g is a homomorphism from (C, γ) to (Y, β). □

Exercise 2.1.14

*For two parallel maps $f, g\colon X \to Y$ between objects X, Y in an arbitrary category $\mathbb{C}$ a **coequaliser** $q\colon Y \to Q$ is a map in a diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] & Q \end{tikzcd}$$

with $q \circ f = q \circ g$ in a 'universal way': for an arbitrary map $h\colon Y \to Z$ with $h \circ f = h \circ g$ tehre is a unique map $k\colon Q \to Z$ with $k \circ q = h$.

1. *An **equalier** in a category $\mathbb{C}$ is a coequaliser in $\mathbb{C}^{\mathrm{op}}$. Formulate explicitly what an equaliser of two parallel maps is.*

2. *Check that in the category* **Sets** *the set Q can be defined as the quotient Y/R, where $R \subseteq Y \times Y$ is the least equivalence relation containing all pairs $(f(x), g(x))$ for $x \in X$.*

3. *Returning to the general case, assume a category $\mathbb{C}$ has coequalisers. Prove that for an arbitrary functor $F\colon \mathbb{C} \to \mathbb{C}$ the associated category of coalgebras* $\mathbf{CoAlg}(F)$ *also has coequalisers, as in $\mathbb{C}$: for two homomorphisms $f, g\colon X \to Y$ between coalgebras $c\colon X \to F(X)$ and $d\colon Y \to F(Y)$ there is by universality an induced coalgebra structure $Q \to F(Q)$ on the coequaliser Q of the underlying maps f, g, yielding a diagram of coalgebras*

$$\begin{pmatrix} F(X) \\ \uparrow c \\ X \end{pmatrix} \mathrel{\substack{\xrightarrow{\ f\ } \\ \xrightarrow[\ g\ ]{}}} \begin{pmatrix} F(Y) \\ \uparrow d \\ Y \end{pmatrix} \xrightarrow{\quad q \quad} \begin{pmatrix} F(Q) \\ \uparrow \\ Q \end{pmatrix}$$

with the appropriate universal property in $\mathbf{CoAlg}(F)$*: for each coalgebra $e\colon Z \to F(Z)$ with homomorphism $h\colon Y \to Z$ satisfying $h \circ f = h \circ g$ there is a unique homomorphism of coalgebras $k\colon Q \to Z$ with $k \circ q = h$.*

Solution.

1. An equaliser of a parallel pair $X \mathrel{\substack{\xrightarrow{f} \\ \xrightarrow[g]{}}} Y$ is a morphism $E \xrightarrow{e} X$ such that both of the following hold:

   (a) we have $fe = ge$; and

   (b) for any morphism $Z \xrightarrow{h} X$ satisfying $fh = gh$ there exists a unique morphism $Z \xrightarrow{k} E$ in $\mathbb{C}$ such that $ek = h$.

2. Fix functions $f, g\colon X \to Y$. Let $R \subseteq Y \times Y$ be the smallest equivalence relation on Y such that $\{ (f(x), g(x)) : x \in X \} \subseteq R$, and define $q\colon Y \to Y/R$ by $q(y) := [y]$ for all $y \in Y$, where $[y]$ denotes the R-equivalence class of $y \in Y$.

   Fix another function $h\colon Y \to Z$ such that $hf = hg$. We need to show that we have the diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "h"'] & Y/R \arrow[d, dashed, "\exists! k"] \\
& & Z
\end{tikzcd}$$

   in **Sets** commuting. We define $k\colon Y/R \to Z$ by $k([y]) := h(y)$ for each R-equivalence class $[y] \in Y/R$. Note that this k is well-defined: if $y, y' \in Y$ are such that yRy' then we can prove by induction on the construction of R (as the reflexive symmetric transitive closure of $\{ (f(x), g(x)) : x \in X \}$) that $h(y) = h(y')$. Then, by construction, $k\colon Y/R \to Z$ is the unique function satisfying $kq = h$.

3. Now suppose that $\mathbb{C}$ has coequalisers. Fix a parallel pair of morphisms $(X, c) \overset{f}{\underset{g}{\rightrightarrows}} (Y, d)$ in $\mathbf{CoAlg}(F)$.

   Let $Y \xrightarrow{q} Q$ be the coequaliser in $\mathbb{C}$ of the parallel pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$. Observe then that

$$\begin{aligned}
(Fq)df &= (Fq)(Ff)c, && \text{since } f \text{ is a homomorphism from } (X,c) \text{ to } (Y,d), \\
&= F(qf)c, && \text{by functoriality of } F, \\
&= F(qg)c, && \text{since } qf = qg, \text{ because } q \text{ is the coequaliser of } f \text{ and } g, \\
&= F(q)F(g)c, && \text{by the functoriality of } F, \\
&= (Fq)dg, && \text{since } g \text{ is also a homomorphism from } (X,c) \text{ to } (Y,d).
\end{aligned}$$

   So there must be a unique morphism $Q \xrightarrow{\alpha} FQ$ in $\mathbb{C}$ such that $\alpha q = (Fq)d$.

$$\begin{tikzcd}
FX \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] & FY \arrow[r, "Fq"'] & FQ \\
X \arrow[u, "c"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[u, "d"'] \arrow[r, "q"] & Q \arrow[u, dashed, "\exists! \alpha"']
\end{tikzcd}$$

   So we have an F-coalgebra structure on Q, namely $Q \xrightarrow{\alpha} FQ$, and the requirement $\alpha q = (Fq)d$ says that q is a homomorphism of F-coalgebras from (Y, d) to (Q, α).

   Now suppose that there is another F-coalgebra $Z \xrightarrow{\beta} FZ$ and a homomorphism $(Y, d) \xrightarrow{h} (Z, \beta)$

such that $hf = hg$. Then there is a unique morphism $Q \xrightarrow{k} Z$ in $\mathbb{C}$ such that $kq = h$.

$$
\begin{tikzcd}
 & & & FZ \\
FX \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] & FY \arrow[r, "Fq"'] \arrow[urr, "Fh"] & FQ \arrow[ur, dashed, "Fk"'] & \\
X \arrow[u, "c"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[u, "d"'] \arrow[r, "q"] \arrow[drr, "h"'] & Q \arrow[u, "\alpha"'] \arrow[dr, dashed, "\exists! k"] & \\
 & & & Z \arrow[uuu, "\beta"']
\end{tikzcd}
$$

We now just need to verify that k is a homomorphism from (Q, α) to (Z, β), i.e. $\beta k = (Fk)\alpha$. We will use the universal property of $Y \xrightarrow{q} Q$ as the coequaliser of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$: we have

$$
\begin{aligned}
\beta h f &= \beta k q f, && \text{since } kq = h, \\
&= \beta k q g, && \text{since } qf = qg, \text{ as } q \text{ coequalises } f \text{ and } g, \\
&= \beta h g, && \text{since } kq = h,
\end{aligned}
$$

and

$$
\begin{aligned}
\beta k q &= \beta h, && \text{since } kq = h, \\
&= (Fh)d, && \text{since } h \text{ is a homomorphism from } (Y, d) \text{ to } (Z, \beta), \\
&= (Fk)(Fq)d, && \text{since } kq = h \text{ and } F \text{ is a functor}, \\
&= (Fk)\alpha q, && \text{since } q \text{ is a homomorphism from } (Y, d) \text{ to } (Q, \alpha).
\end{aligned}
$$

The equalities to take away from the second calculation above are

$$\beta k q = \beta h = (Fk)\alpha q.$$

By the uniqueness clause in the universal property of coequalisers, we must have $\beta k = (Fk)\alpha$. $\square$

2.2 Polynomial Functors and Their Coalgebras

Exercise 2.2.1
Check that a polynomial functor which does not contain the identity functor is constant.

Solution. This follows by induction on the complexity of polynomial functors. ☐

Exercise 2.2.2
Describe the kind of trees that can arise as behaviours of coalgebras:

1. $S \to A + (A \times S)$.
2. $S \to A + (A \times S) + (A \times S \times S)$.

Solution.

1. A coalgebra $S \to A + (A \times S)$ can give rise to any of the following kinds of trees:

$$\begin{tikzcd}
a_0 & a_0 \arrow[d, dash] & a_0 \arrow[d, dash] & a_0 \arrow[d, dash] & & a_0 \arrow[d, dash] & & a_0 \arrow[d, dash] \\
& a_1 & a_1 \arrow[d, dash] & a_1 \arrow[d, dash] & \cdots & a_1 \arrow[d, dash] & \cdots & a_1 \arrow[d, dash] \\
& & a_2 & a_2 \arrow[d, dash] & & \vdots \arrow[d, dash] & & a_2 \arrow[d, dash] \\
& & & a_3 & & a_{n-1} \arrow[d, dash] & & a_3 \arrow[d, dash] \\
& & & & & a_n & & \vdots
\end{tikzcd}$$

   That is, trees where every node has at most one successor.

2. A coalgebra $S \to A + (A \times S) + (A \times S \times S)$ gives rise to a tree where every node has at most two successors. The tree

$$\begin{tikzcd}
& a_0 \arrow[dl] \arrow[dr] & & \\
a_1 \arrow[d] & & a'_1 \arrow[d] & \\
a_2 & & a'_2 \arrow[d] & \\
& & a_3 \arrow[dl] \arrow[dr] & \\
& a_4 & & a'_4 \arrow[d] \\
& & & \vdots
\end{tikzcd}$$

   is an example of such a tree. ☐

Exercise 2.2.3
Check, using Exercise 2.1.10, that non-deterministic automata $X \to \mathcal{P}(X)^A \times 2$ can equivalently be described as transition systems $X \to \mathcal{P}(1 + (A \times X))$. Work out the correspondence in detail.

Solution. Write $1 = \{*\}$ and $2 = \{0, 1\}$. For a function $f\colon X \to \mathcal{P}(X)^A \times 2$, define $\varphi_f\colon X \to \mathcal{P}(1 + (A \times X))$ by

$$\varphi_f(x) := \begin{cases} \{*\} \cup \{\, (a, z) \in A \times X : z \in h(a) \,\}, & \text{if } f(x) = (h, 1) \text{ for some function } h\colon A \to \mathcal{P}(X), \\ \{\, (a, z) \in A \times X : z \in h(a) \,\}, & \text{if } f(x) = (h, 0) \text{ for some function } h\colon A \to \mathcal{P}(X), \end{cases}$$

for all $x \in X$. For a function $g\colon X \to \mathcal{P}(1 + (A \times X))$, define $\psi_g\colon X \to \mathcal{P}(X)^A \times 2$ by

$$\psi_g(x) := \begin{cases} (\lambda a \in A.\{\, z \in X : (a, z) \in g(x) \,\},\ 1), & \text{if } * \in g(x), \\ (\lambda a \in A.\{\, z \in X : (a, z) \in g(x) \,\},\ 0), & \text{if } * \notin g(x), \end{cases}$$

for all $x \in X$. Then $\psi_{\varphi_f} = f$ and $\varphi_{\psi_g} = g$ for all functions $f\colon X \to \mathcal{P}(X)^A \times 2$ and functions $g\colon X \to \mathcal{P}(1 + (A \times X))$. $\square$

Exercise 2.2.4
Describe the arity $\#$ for the functors

1. $X \mapsto B + (X \times A \times X)$.
2. $X \mapsto A_0 \times X^{A_1} \times (X \times X)^{A_2}$, *for finite sets* A_1, A_2.

Solution.

1. Define an arity $\#\colon A + B \to \mathbb{N}$ by $\#a := 2$, for all $a \in A$, and $\#b := 0$, for all $b \in B$. Then the associated arity functor $F_\#\colon \mathbf{Sets} \to \mathbf{Sets}$ satisfies

$$\begin{aligned} F_\# X &= \coprod_{i \in A+B} X^{\#i} \\ &= \coprod_{b \in B} X^{\#b} + \coprod_{a \in A} X^{\#a} \\ &= \coprod_{b \in B} X^0 + \coprod_{a \in A} X^2 \\ &\cong B + \coprod_{a \in A} (X \times X) \\ &\cong B + (X \times A \times X), \end{aligned}$$

for all $X \in \mathsf{Obj}(\mathbf{Sets})$.

2. Define an arity $\#\colon A_0 \to \mathbb{N}$ by $\#i := |A_1| + |A_2| + |A_2|$ for all $i \in A_0$. Then the associated arity functor $F_\#\colon \mathbf{Sets} \to \mathbf{Sets}$ satisfies

$$\begin{aligned} F_\# X &= \coprod_{i \in A_0} X^{\#i} \\ &= \coprod_{i \in A_0} X^{|A_1| + |A_2| + |A_2|} \end{aligned}$$

$$
\begin{aligned}
&\cong \coprod_{i \in A_0} (X^{A_1} \times (X^{A_2} \times X^{A_2})) \\
&\cong \coprod_{i \in A_0} (X^{A_1} \times (X \times X)^{A_2}) \\
&\cong A_0 \times X^{A_1} \times (X \times X)^{A_2},
\end{aligned}
$$

for all $X \in \mathsf{Obj}(\mathbf{Sets})$. $\square$

Exercise 2.2.5

Check that finite arity functors correspond to simple polynomial functors in the construction of which all constant functors $X \mapsto A$ and exponents X^A have finite sets A.

Solution. Let finSPF be this class of simple polynomial functors. Clearly finite arity functors are in finSPF. The proof that all functors in finSPF are of finite arity proceeds by induction on the structure of functors in finSPF, much along the lines of the proof of Proposition 2.2.3. $\square$

Exercise 2.2.6

Consider an indexed collection of sets $(A_i)_{i \in I}$ and define the associated 'dependent' polynomial functor $\mathbf{Sets} \to \mathbf{Sets}$ *by*

$$X \mapsto \coprod_{i \in I} X^{A_i} = \{ (i, f) \mid i \in I \wedge f \colon A_i \to X \}.$$

1. *Prove that we get a functor in this way; obviously by Proposition 2.2.3, each polynomial functor is of this form, for a finite set A_i.*

2. *Check that all simple polynomial functors are dependent — by finding suitable collections $(A_i)_{i \in I}$ for each of them.*

(These functors are studied as 'containers' in the context of so-called W-types in dependent type theory for well-founded trees; see for instance Abbott, Altenkirch, and Ghani (2003a), Abbott, Alternkirch, and Ghani (2005), and Moerdijk and Palmgreen (2000).

Solution.

1. The functor $X \mapsto \coprod_{i \in I} X^{A_i}$ maps a function $g \colon X \to Y$ to the function

$$\coprod_{i \in I} X^{A_i} \ni (i, f) \mapsto (i, gf) \in \coprod_{i \in I} Y^{A_i}.$$

   Functoriality follows from the associativity of function composition.

2. We induct on the complexity of simple polynomial functors.

   The identity functor $\mathrm{id}_{\mathbf{Sets}}$ is the dependent polynomial functor associated with the collection $(A_i)_{i \in I} = (1)_{i \in 1}$.

   The constant functor at $A \in \mathsf{Obj}(\mathbf{Sets})$ is the dependent polynomial functor associated with the collection $(A_i)_{i \in I} = (0)_{i \in A}$.

   If F and G are both simple polynomial functors and we inductively have $FX = \coprod_{i \in I} X^{A_i}$ and $GX = \coprod_{j \in J} X^{B_j}$ for all $X \in \mathsf{Obj}(\mathbf{Sets})$, then $(F \times G)(X) = \coprod_{(i,j) \in I \times J} X^{A_i + B_j}$ for all $X \in \mathsf{Obj}(\mathbf{Sets})$.

   If $(F_i)_{i \in I}$ is an I-indexed collection of simple polynomial functors, say with $F_i X = \coprod_{j \in J_i} X^{A_{i,j}}$ for all $X \in \mathsf{Obj}(\mathbf{Sets})$, then $(\coprod_{i \in I} F_i)(X) = \coprod_{i \in I} \coprod_{j \in J_i} X^{A_{i,j}} = \coprod_{(i,j) \in \coprod_{i \in I} J_i} X^{A_{i,j}}$. $\square$

Exercise 2.2.7
Recall from (2.13) and (2.14) that the powerset functor $\mathcal{P}$ can be described both as a covariant functor $\mathbf{Sets} \to \mathbf{Sets}$ *and as a contravariant one* $2^{(-)} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$. *In the definition of Kripke polynomial functors we use the powerset $\mathcal{P}$ covariantly. The functor $\mathcal{N} = \mathcal{P}\mathcal{P} \colon \mathbf{Sets} \to \mathbf{Sets}$ is obtained by using the contravariant powerset functor twice — yielding a covariant, but not Kripke polynomial functor. Coalgebras of this so-called neighbourhood functor are used in Scott (1970) and Montague (1970) as models of a special modal logic (see also Hansen, Kupke, and Pacuit (2009) and Hansen, Kupke, and Leal (2014) for the explicitly coalgebraic view).*

1. *Describe the action $\mathcal{N}(f) \colon \mathcal{N}(X) \to \mathcal{N}(Y)$ of a function $f \colon X \to Y$.*

2. *Try to see a coalgebra $c \colon X \to \mathcal{N}(X)$ as the setting of a two-player game, with the first player's move in state $x \in X$ given by a choice of a subset $U \in c(x)$ and the second player's reply by a choice of successor state $x' \in U$.*

Solution.

1. Fix a function $f \colon X \to Y$. The function $\mathcal{P}f \colon \mathcal{P}Y \to \mathcal{P}X$, where $\mathcal{P} \colon \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ denotes the contravariant powerset functor, is given by

$$(\mathcal{P}f)(B) \coloneqq f^{-1}(B) = \{\, x \in X : f(x) \in B \,\},$$

for all $B \in \mathcal{P}Y$. So the function $\mathcal{N}f \colon \mathcal{N}X \to \mathcal{N}Y$ is given by

$$\begin{aligned}(\mathcal{N}f)(\mathscr{A}) &= (\mathcal{P}\mathcal{P}f)(\mathscr{A}) \\ &= (\mathcal{P}f)^{-1}(\mathscr{A}) \\ &= \{\, B \in \mathcal{P}Y : (\mathcal{P}f)(B) \in \mathscr{A} \,\} \\ &= \{\, B \in \mathcal{P}Y : f^{-1}(B) \in \mathscr{A} \,\},\end{aligned}$$

for $\mathscr{A} \in \mathcal{N}X = \mathcal{P}\mathcal{P}X$.

2. Just read the question as described. □

Exercise 2.2.8

1. *Notice that the behaviour function* $\mathrm{beh} \colon S \to B^{A^\star}$ *from (2.23) for a deterministic automaton satisfies:*

$$\begin{aligned}\mathrm{beh}(x)(\langle\rangle) &= \epsilon(x) \\ &= b && \textit{where } x \downarrow b \\ \mathrm{beh}(x)(a \cdot \sigma) &= \mathrm{beh}(\delta(x)(a))(\sigma) \\ &= \mathrm{beh}(x')(\sigma) && \textit{where } x \xrightarrow{a} x'.\end{aligned}$$

2. *Consider a homomorphism $f \colon X \to Y$ of coalgebras/deterministic automata from $X \to X^A \times B$ and $Y \to Y^A \times B$ and prove that for all $x \in X$,*

$$\mathrm{beh}_2(f(x)) = \mathrm{beh}_1(x)$$

Solution. Recall that $A^\star$ denotes the set of all finite sequences of elements of A. Furthermore, recall that for a coalgebra $S \xrightarrow{\langle \delta, \epsilon \rangle} S^A \times B$, the behaviour function $\mathrm{beh}\colon S \to B^{A^\star}$ is defined by

$$\mathrm{beh}(x) := \lambda\sigma \in A^\star.(\epsilon(\delta^*(x,\sigma))), \quad \text{for all } x \in S,$$

where $\delta^*\colon S \times A^\star \to S$ is defined by

$$\delta^*(x,\tau) := \begin{cases} x, & \text{if } \tau = \langle\rangle, \\ \delta^*(\delta(x)(a),\sigma), & \text{if } \tau = a \cdot \sigma \text{ for some } a \in A \text{ and } \sigma \in A^\star, \end{cases}$$

for all $(x,\tau) \in S \times A^\star$.

1. Fix $x \in S$. Suppose that $x \downarrow b$ and $x \xrightarrow{a} x'$, i.e. $\epsilon(x) = b$ and $x' = \delta(x)(a)$. Then

$$\begin{aligned} \mathrm{beh}(x)(\langle\rangle) &= \epsilon(\delta^*(x,\langle\rangle)) \\ &= \epsilon(x) \\ &= b. \end{aligned}$$

   Moreover, for $(a,\sigma) \in S \times A^\star$, we have

$$\begin{aligned} \mathrm{beh}(x)(a \cdot \sigma) &= \epsilon(\delta^*(x, a \cdot \sigma)) \\ &= \epsilon(\delta^*(\delta(x)(a),\sigma)) \\ &= \mathrm{beh}(\delta(x)(a))(\sigma) \\ &= \mathrm{beh}(x')(\sigma). \end{aligned}$$

2. Now fix coalgebras $X \xrightarrow{\langle \delta_1, \varepsilon_1 \rangle} X^A \times B$ and $Y \xrightarrow{\langle \delta_2, \varepsilon_2 \rangle} Y^A \times B$, and fix a homomorphism of coalgebras $f\colon (X, \langle \delta_1, \varepsilon_1 \rangle) \to (Y, \langle \delta_2, \varepsilon_2 \rangle)$. We have the commuting diagram

$$\begin{tikzcd} X^A \times B \arrow[r, "f^A \times \mathrm{id}_B"] & Y^A \times B \\ X \arrow[u, "\langle \delta_1, \varepsilon_1 \rangle"] \arrow[r, "f"'] & Y \arrow[u, "\langle \delta_2, \varepsilon_2 \rangle"'] \end{tikzcd}$$

   in **Sets**, where the notation f^A was introduced in our solution to Exercise 2.1.8.3. This diagram commuting says that $(\delta_2(f(x)), \varepsilon_2(f(x)) = (f^A(\delta_1(x)), \varepsilon_1(x))$ for all $x \in X$. Also, the associated behaviour functions $\mathrm{beh}_1\colon X \to B^{A^\star}$ and $\mathrm{beh}_2\colon Y \to B^{A^\star}$ satisfy

$$\mathrm{beh}_1(x)(\sigma) = \varepsilon_1(\delta_1^*(x,\sigma)) \quad \text{and} \quad \mathrm{beh}_2(f(x))(\sigma) = \varepsilon_2(\delta_2^*(f(x),\sigma))$$

   for all $x \in X$ and $\sigma \in A^\star$. We will prove, by induction on the length of σ, that $\mathrm{beh}_1(x)(\sigma) = \mathrm{beh}_2(f(x))(\sigma)$ for all $x \in X$ and $\sigma \in A^\star$. This will yield $\mathrm{beh}_1(x) = \mathrm{beh}_2(f(x))$ for all $x \in X$.

   For the base case of the induction, we have

$$\mathrm{beh}_1(x)(\langle\rangle) = \varepsilon_1(x) = \varepsilon_2(f(x)) = \mathrm{beh}_2(f(x)(\langle\rangle),$$

   for all $x \in X$, using Exercise 2.2.8.1 and the fact that f is a homomorphism of coalgebras.

Now, suppose inductively that a given $\sigma \in A^\star$ satisfies $\mathrm{beh}_1(x)(\sigma) = \mathrm{beh}_2(f(x))(\sigma)$ for all $x \in X$. Using Exercise 2.2.8.1 again, we have

$$\begin{aligned}
\mathrm{beh}_1(x)(a \cdot \sigma) &= \mathrm{beh}_1\big(\delta_1(x)(a)\big)(\sigma) \\
&= \mathrm{beh}_2\big(f(\delta_1(x)(a))\big)(\sigma), && \text{by the inductive hypothesis,} \\
&= \mathrm{beh}_2\big(f^A(\delta_1(x))(a)\big)(\sigma), && \text{by definition of } f^A \colon X^A \to Y^A, \\
&= \mathrm{beh}_2\big(\delta_2(f(x))(a)\big)(\sigma), && \text{since } f \text{ is a homomorphism of coalgebras,} \\
&= \mathrm{beh}_2(f(x))(a \cdot \sigma),
\end{aligned}$$

for all $a \in A$. □

Exercise 2.2.9

Check that the iterated transition function $\delta^* \colon S \times A^\star \to S$ of a deterministic automaton is a monoid action — see Exercise 1.4.1 — for the free monoid structure on $A^\star$ from Exercise 1.4.4.

Solution. The identity element of the monoid $A^\star$ is the empty sequence $\langle\rangle$, and we indeed have $\delta^*(x, \langle\rangle) = x$ for all $x \in S$. We now prove that

$$\delta^*(x, \sigma \cdot \tau) = \delta^*(\delta^*(x, \sigma), \tau)$$

for all $x \in S$ and finite sequences $\sigma, \tau \in A^\star$. This will be proven by induction on the length of σ.

If $\sigma = \langle\rangle$, then

$$\delta^*(x, \langle\rangle \cdot \tau) = \delta^*(x, \tau) = \delta^*(\delta^*(x, \langle\rangle), \tau),$$

for all $x \in S$ and $\tau \in A^\star$, where we have used that $x = \delta^*(x, \langle\rangle)$ in the second equality. Now suppose inductively that a given $\sigma \in A^\star$ satisfies $\delta^*(x, \sigma \cdot \tau) = \delta^*(\delta^*(x, \sigma), \tau)$ for all $x \in S$ and $\tau \in A^\star$. Then, for any $a \in A$, we have

$$\begin{aligned}
\delta^*(x, (a \cdot \sigma) \cdot \tau) &= \delta^*(x, a \cdot (\sigma \cdot \tau)) \\
&= \delta^*(\delta(x)(a), \sigma \cdot \tau), && \text{by definition of } \delta^*, \\
&= \delta^*\big(\delta^*(\delta(x)(a), \sigma), \tau\big), && \text{by the inductive hypothesis,} \\
&= \delta^*(\delta^*(x, a \cdot \sigma), \tau), && \text{by definition of } \delta^*,
\end{aligned}$$

as desired. □

Exercise 2.2.10

Note that a function space S^S carries a monoid structure given by composition. Show that the iterated transition function δ^* for a deterministic automaton, considered as a monoid homomorphism $A^\star \to S^S$, is actually obtained from δ by freeness of $A^\star$ — as described in Exercise 1.4.4.

Solution. We know that we can consider the transition function δ as a function from A to S^S. This lets us consider the iterated transition function as a function from $A^\star$ to S^S, given by

$$\delta^*(\tau) \coloneqq \begin{cases} \mathrm{id}_S, & \text{if } \tau = \langle\rangle, \\ \delta^*(\sigma) \circ \delta(a), & \text{if } \tau = a \cdot \sigma \text{ for some } a \in A \text{ and } \sigma \in A^\star, \end{cases}$$

for all $\tau \in A^\star$. The associativity of function composition implies that $\delta^* \colon A^\star \to S^S$ is a monoid homomorphism. So, the diagram

$$\begin{tikzcd}
 & S^S \\
A \arrow[ur, "\delta"] \arrow[r, "a \mapsto \langle a \rangle"'] & A^\star \arrow[u, "\delta^*"']
\end{tikzcd}$$

in **Sets** commutes. By Exercise 1.4.4.3, the homomorphism $\delta^*\colon A^\star \to S^S$ must be the unique homomorphism obtained from δ by freeness of $A^\star$. □

Exercise 2.2.11
Consider a very simple differential equation of the form $df/dy = -Cf$, where $C \in \mathbb{R}$ is a fixed positive constant. The solution is usually described as $f(y) = f(0)\cdot e^{-Cy}$. Check that it can be described as a monoid action $\mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, namely $(x, y) \mapsto xe^{-Cy}$, where $\mathbb{R}_{\geq 0}$ is the monoid of non-negative real numbers with addition $+, 0$.

Solution. Let $\mu\colon \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by $\mu(x, y) \coloneqq xe^{-Cy}$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$. Then, for all $x \in \mathbb{R}$, we have

$$\mu(x, 0) = xe^{-C\cdot 0} = xe^0 = x.$$

Furthermore, for all $x \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}_{\geq 0}$, we have

$$\mu(x, y_1 + y_2) = xe^{-C(y_1+y_2)} = \left(xe^{-Cy_2}\right)e^{-Cy_1} = \mu(\mu(x, y_2), y_1).$$

So μ describes a monoid action of $\mathbb{R}_{\geq 0}$ on $\mathbb{R}$. □

Exercise 2.2.12
Let **Vect** *be the category with finite-dimensional vector spaces over the real numbers $\mathbb{R}$ (or some other field) as objects, and with linear transformations between them as morphisms. This exercise describes the basics of linear dynamical systems, in analogy with deterministic automata. It does require some basic knowledge of vector spaces.*

1. *Prove that the product $V \times W$ of (the underlying sets of) two vector spaces V and W is at the same time a product and a coproduct in* **Vect** *— the same phenomenon as in the category of monoids; see Exercise 2.1.6. Show also that the singleton space 1 is both an initial and a final object. And notice that an element x in a vector space V may be identified with a linear map $\mathbb{R} \to V$.*

2. *A* ***linear dynamical system*** *(Kálmán, Falb, and Arbib, 1969) consists of three vector spaces: S for the state space, A for the inpur and B for the output, together with three linear transformations: an input map $G\colon A \to S$, a dynamics $F\colon S \to S$ and an output map $H\colon S \to B$. Note how the first two maps can be combined via cotupling into one transition function $S \times A \to S$, as used for deterministic automata. Because of the possibility of decomposing the transition function in this linear case into two maps $A \to S$ and $S \to S$, these systems are called decomposable by Arbib and Manes (1974). (But this transition function $S \times A \to S$ is not bilinear (i.e. linear in each argument separately), so it does not give rise to a map $S \to S^A$ to the vector space S^A of linear transformations from A to S. Hence we do not have a purely coalgebraic description $S \to S^A \times B$ in this linear setting.)*

3. *For a vector space A, consider, in the notation of Arbib and Manes (1974), the subset of infinite sequences:*

$$A^\S = \{\, \alpha \in A^{\mathbb{N}} \mid \textit{only finitely many } \alpha(n) \textit{ are non-zero} \,\}.$$

Equip the set $A^\S$ with a vector space structure, such that the insertion map $\mathsf{in}\colon A \to A^\S$, defined as $\mathsf{in}(a) = (a, 0, 0, 0, \dots)$, and shift map $\mathsf{sh}\colon A^\S \to A^\S$, given as $\mathsf{sh}(\alpha) = (0, \alpha(0), \alpha(1), \dots)$, are linear transformations. (This vector space $A^\S$ may be understood as the space of polynomials over A in one variable. It can be defined as the infinite coproduct $\coprod_{n\in\mathbb{N}} A$ of $\mathbb{N}$-copies of A — which is also called a copower and written as $\mathbb{N} \cdot A$; see Mac Lane (1978, III, 3). It is the analogue in **Vect** *of the set of finite sequences $B^\star$ for $B \in$* **Sets**. *This will be made precise in Exercise 2.4.8.)*

4. Consider a linear dynamical system $A \xrightarrow{G} S \xrightarrow{F} S \xrightarrow{H} B$ as above and show that the analogue of the behaviour $A^\star \to B$ for deterministic automata (see also Arbib and Manes (1975b, 6.3)) is the linear map $A^\S \to B$ defined as

$$(a_0, a_1, \ldots, a_n, 0, 0, \ldots) \mapsto \sum_{i \le n} HF^i G a_i.$$

This is the standard behaviour formula for linear dynamical systems; see e.g. Kálmán, Falb, and Arbib (1969) and Arbib and Manes (1980). (This behaviour map can be understood as starting from the 'default' initial state $0 \in S$. If one wishes to start from an arbitrary initial state $x \in S$, one gets the formula

$$(a_0, a_1, \ldots, a_n, 0, 0 \ldots) \mapsto HF^{(n+1)}x + \sum_{i \le n} HF^i G a_i.$$

It is obtained by consecutively modifying the state x with inputs $a_n, a_{n-1}, \ldots, a_0$.)

Solution.

1. Our solution to Exercise 2.1.6 can be adapted to obtain a proof that finite products and coproducts in **Vect** coincide.

   For $V \in \mathbf{Vect}$, any vector $x \in V$ can be identified with the linear map $\varphi_x \colon \mathbb{R} \to V$ given by $\varphi_x(k) \coloneqq kx$. As $\mathbb{R}$ can be seen as a vector space over itself, the set $\hom_{\mathbf{Vect}}(\mathbb{R}, V)$ can be given a vector space structure. The mapping $V \ni x \mapsto \varphi_x \in \hom_{\mathbf{Vect}}(\mathbb{R}, V)$ is then a linear isomorphism.

2. This is just reading. There is no exercise here.

3. For $\alpha, \beta \in A^\S$, we define $\alpha + \beta \in A^\S$ by $(\alpha + \beta)(n) \coloneqq \alpha(n) + \beta(n)$ for all $n \in \mathbb{N}$. Also, for $\alpha \in A^\S$ and $k \in \mathbb{R}$, we define $k\alpha \in A^\S$ by $(k\alpha)(n) \coloneqq k \cdot \alpha(n)$ for all $n \in \mathbb{N}$. Together with the vector $(0, 0, 0, \ldots)$, these equip $A^\S$ with a vector space structure such that the insertion and shift maps are linear maps.

4. Motivated by Equation (2.22) for deterministic automata, we define $\delta^* \colon A^\S \to S$ by

$$\delta^*(\alpha) \coloneqq \begin{cases} 0, & \text{if } \alpha = (0, 0, 0, \ldots), \\ G(\alpha(0)) + F\Big(\delta^*\big((\alpha(1), \alpha(2), \alpha(3), \ldots)\big)\Big), & \text{otherwise,} \end{cases}$$

   for all $\alpha \in A^\S$. This is well-defined since each $\alpha \in A^\S$ only has finitely many non-zero entries. Then, motivated by Equation (2.23), we define $\mathrm{beh} \colon A^\S \to B$ by $\mathrm{beh}(\alpha) \coloneqq H(\delta^*(\alpha))$ for all $\alpha \in A^\S$. This agrees the formula as claimed. □

2.3 Final Coalgebras

Exercise 2.3.1

Check that a final coalgebra of a monotone endofunction $f\colon X \to X$ on a poset X, considered as a functor, is nothing but a greatest fixed point. (See also Exercise 1.3.5.)

Solution. In a poset $(X, \leq)$, the only isomorphisms are the identity morphisms. So there is only one final coalgebra (up to equality, not just up to isomorphism), if it exists.

Suppose $x \in X$ is the (carrier of the) final f-coalgebra. Then x is a fixed point of f, by Lambek's lemma (see Lemma 2.3.3). Furthermore, letting $A \coloneqq \{\, y \in X : y \leq f(y) \,\}$ be the set of all f-coalgebras, we have that $x = \max A$. As every fixed point of f is also an f-coalgebra, this point x must be the greatest fixed point of f. $\square$

Exercise 2.3.2

For arbitrary sets A, B, consider the (simple polynomial) functor $X \mapsto (X \times B)^A$. Coalgebras $X \to (X \times B)^A$ of this functor are often called Mealy machines.

1. *Check that Mealy machines can equivalently be described as deterministic automata $X \to X^A \times B^A$ and that the final Mealy machine is B^{A^+}, by Proposition 2.3.5, where $A^+ \hookrightarrow A^\star$ is the subset of non-empty finite sequences. Describe the final coalgebra structure $B^{A^+} \to (B^{A^+} \times B)^A$ explicitly.*

2. *Fix a set A and consider the final coalgebra $A^{\mathbb{N}}$ of the stream functor $A \times (-)$. Define by finality an 'alternation' function $\mathsf{alt}\colon A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$, such that*

$$\mathsf{alt}(a_0 a_1 \cdots,\, b_0 b_1 b_2 \cdots) = a_0 b_1 a_2 b_3 \cdots .$$

   Prove by coinduction

$$\mathsf{alt}\big(\sigma, \mathsf{alt}(\tau_1, \rho)\big) = \mathsf{alt}\big(\sigma, \mathsf{alt}(\tau_2, \rho)\big).$$

   Thus in such a combination the middle argument is irrelevant.

3. *Consider the set Z of so-called causal stream functions, given by*

$$Z = \Big\{ \psi\colon A^{\mathbb{N}} \to B^{\mathbb{N}} \;\Big|\; \forall n \in \mathbb{N}. \forall \alpha, \alpha' \in A^{\mathbb{N}}. \\ \big(\forall i \leq n. \alpha(i) = \alpha'(i)\big) \Rightarrow \psi(\alpha)(n) = \psi(\alpha')(n) \Big\}.$$

   For such a causal stream function ψ, the output $\psi(\alpha)(n) \in B$ is thus determined by the first $n+1$ elements $\alpha(0), \ldots, \alpha(n) \in A$. Prove that Z yields an alternative description of the final Mealy automaton, via the structure map $\zeta\colon Z \to (Z \times B)^A$ given by

$$\zeta(\psi)(a) = (\, \lambda \alpha \in A^{\mathbb{N}}. \lambda n. \psi(a \cdot \alpha)(n+1),\; \psi(\lambda n \in \mathbb{N}. a)(0) \,)$$

   where $a \cdot \alpha$ is prefixing to $\alpha \in A^{\mathbb{N}}$, considered as an infinite sequence.

For more information on Mealy machines, see Bonsangue, Rutten, and Silva (2008) and Hansen and Rutten (2010).

Solution.

1. We can uniquely associate to each morphism $X \to (X \times B)^A$ a morphism $X \to X^A \times B^A$, simply because $(X \times B)^A \cong X^A \times B^A$. Recall that $A^\star$ is the set of all finite sequences of elements

of A. By Proposition 2.3.5, the final Mealy machine, i.e. the final coalgebra of the functor $(-)^A \times B^A \colon \mathbf{Sets} \to \mathbf{Sets}$, is the coalgebra $\zeta \colon (B^A)^{A^\star} \to \big((B^A)^{A^\star}\big)^A \times B^A$ defined by

$$\zeta(\varphi) \coloneqq \big(\lambda a \in A.\lambda\sigma \in A^\star.\varphi(a \cdot \sigma),\, \varphi(\langle\rangle)\big),$$

for all functions $\varphi \colon A^\star \to B^A$. By currying and using the fact that $A^\star \times A \cong A^+$, we can equivalently describe this coalgebra $\zeta \colon (B^A)^{A^\star} \to \big((B^A)^{A^\star}\big)^A \times B^A$ as the coalgebra $\zeta' \colon B^{A^+} \to (B^{A^+})^A \times B^A$ defined by

$$\zeta'(\varphi) \coloneqq \big(\lambda a \in A.\lambda\sigma \in A^+.\varphi(a \cdot \sigma),\, \lambda a \in A.\varphi(\langle a\rangle)\big),$$

for all functions $\varphi \colon A^+ \to B$. This yields the coalgebra $\zeta'' \colon B^{A^+} \to (B^{A^+} \times B)^A$ (of the functor $((-) \times B)^A \colon \mathbf{Sets} \to \mathbf{Sets}$) defined by

$$\zeta''(\varphi) \coloneqq \lambda a \in A.\big(\lambda\sigma \in A^+.\varphi(a \cdot \sigma),\, \varphi(\langle a\rangle)\big),$$

for all functions $\varphi \colon A^+ \to B$. This $\zeta'' \colon B^{A^+} \to (B^{A^+} \times B)^A$ is indeed the final Mealy machine: given any other Mealy machine $c \colon X \to (X \times B)^A$, the unique function $f \colon X \to B^{A^+}$ making the diagram

$$\begin{tikzcd}
(X \times B)^A \arrow[r, dashed, "(f \times \mathrm{id}_B)^A"] & \big(B^{A^+} \times B\big)^A \\
X \arrow[u, "c"] \arrow[r, dashed, "\exists! f"'] & B^{A^+} \arrow[u, "\cong"', "\zeta''"]
\end{tikzcd}$$

in **Sets** commute must satisfy

$$f(x) = \lambda\sigma \in A^+.\begin{cases} \pi_2(c(x)(a)), & \text{if } \sigma = \langle a\rangle \text{ for some } a \in A, \\ f\big(\pi_1(c(x)(a))\big)(\tau), & \text{if } \sigma = a \cdot \tau \text{ for some } a \in A \text{ and } \tau \in A^+, \end{cases}$$

for all $x \in X$, where $X \xleftarrow{\pi_1} X \times B \xrightarrow{\pi_2} B$ are the relevant projection functions.

2. Again, by Proposition 2.3.5, the final coalgebra $\gamma \colon A^{\mathbb{N}} \to A \times A^{\mathbb{N}}$ of the functor $A \times (-) \colon \mathbf{Sets} \to \mathbf{Sets}$ is defined by

$$\gamma(\sigma) \coloneqq \big(\sigma_0,\, (\sigma_1, \sigma_2, \sigma_3, \dots)\big)$$

for all streams $\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3, \dots) \in A^{\mathbb{N}}$. Now define a coalgebra $c \colon A^{\mathbb{N}} \times A^{\mathbb{N}} \to A \times (A^{\mathbb{N}} \times A^{\mathbb{N}})$ by

$$c(\sigma, \tau) \coloneqq \Big(\sigma_0,\ \big((\tau_1, \tau_2, \tau_3, \dots), (\sigma_1, \sigma_2, \sigma_3, \dots)\big)\Big),$$

for all streams $\sigma, \tau \in A^{\mathbb{N}}$ with $\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3, \dots)$ and $\tau = (\tau_0, \tau_1, \tau_2, \tau_3, \dots)$. In other words, writing $\gamma = \langle \mathrm{head}, \mathrm{tail}\rangle$, where $\mathrm{head} \colon A^{\mathbb{N}} \to A$ and $\mathrm{tail} \colon A^{\mathbb{N}} \to A^{\mathbb{N}}$ are defined by

$$\mathrm{head}(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \dots) \coloneqq \sigma_0 \quad \text{and} \quad \mathrm{tail}(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \dots) \coloneqq (\sigma_1, \sigma_2, \sigma_3, \dots)$$

for all $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \ldots \in A$, we have $c(\sigma, \tau) = (\mathrm{head}(\sigma),\ (\mathrm{tail}(\tau), \mathrm{tail}(\sigma)))$ for all $(\sigma, \tau) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$. Then we have the commuting diagram

$$\begin{tikzcd}
A \times (A^{\mathbb{N}} \times A^{\mathbb{N}}) \arrow[r, dashed, "\mathrm{id}_A \times \mathsf{alt}"] & A \times A^{\mathbb{N}} \\
A^{\mathbb{N}} \times A^{\mathbb{N}} \arrow[u, "c"] \arrow[r, dashed, "\exists! \mathsf{alt}"'] & A^{\mathbb{N}} \arrow[u, "\cong"', "\gamma = \langle \mathrm{head}, \mathrm{tail}\rangle"]
\end{tikzcd}$$

in **Sets**. This function $\mathsf{alt}\colon A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$ is our desired alternation function: indeed, the commuting diagram above implies that

$$\begin{aligned}
\mathsf{alt}(\sigma,\tau) &= \mathrm{head}(\sigma)\cdot \mathsf{alt}(\mathrm{tail}(\tau),\mathrm{tail}(\sigma)) \\
&= \mathrm{head}(\sigma)\cdot \mathrm{head}(\mathrm{tail}(\tau))\cdot \mathsf{alt}(\mathrm{tail}(\mathrm{tail}(\sigma)),\ \mathrm{tail}(\mathrm{tail}(\tau))) \\
&= \sigma_0\cdot\tau_1\cdot \mathsf{alt}\big((\sigma_2,\sigma_3,\dots),\ (\tau_2,\tau_3,\dots)\big)
\end{aligned}$$

for all $(\sigma,\tau) = \big((\sigma_0,\sigma_1,\sigma_2,\sigma_3,\dots),\ (\tau_0,\tau_1,\tau_2,\tau_3,\dots)\big) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$.

Now let us show that $\mathsf{alt}(\sigma,\mathsf{alt}(\tau,\rho)) = \mathsf{alt}(\sigma,\mathsf{alt}(\tau',\rho))$ for all $\sigma,\tau,\tau',\rho \in A^{\mathbb{N}}$. In fact, we will show that an even stronger result holds: $\mathsf{alt}(\sigma,\mathsf{alt}(\tau,\rho)) = \mathsf{alt}(\sigma,\rho) = \mathsf{alt}(\mathsf{alt}(\sigma,\tau),\rho)$ for all $\sigma,\tau,\rho \in A^{\mathbb{N}}$. Define a set $R \subseteq A^{\mathbb{N}} \times A^{\mathbb{N}}$ to be

$$R \coloneqq \Big\{\ \big(\mathsf{alt}(\sigma,\mathsf{alt}(\tau,\rho)),\ \mathsf{alt}(\sigma,\rho)\big)\ :\ \sigma,\tau,\rho \in A^{\mathbb{N}}\Big\} \cup \Big\{\ \big(\mathsf{alt}(\mathsf{alt}(\sigma,\tau),\rho),\ \mathsf{alt}(\sigma,\rho)\big)\ :\ \sigma,\tau,\rho \in A^{\mathbb{N}}\Big\},$$

and define a function $e\colon R \to A \times R$ by

$$e\big(\mathsf{alt}(\sigma,\mathsf{alt}(\tau,\rho)),\ \mathsf{alt}(\sigma,\rho)\big) \coloneqq \Big(\mathrm{head}(\sigma),\ \big(\mathsf{alt}(\mathsf{alt}(\mathrm{tail}(\rho),\mathrm{tail}(\tau)),\mathrm{tail}(\sigma)),\ \mathrm{alt}(\mathrm{tail}(\rho),\mathrm{tail}(\sigma))\big)\Big)$$

and

$$e\big(\mathsf{alt}(\mathsf{alt}(\sigma,\tau),\rho),\ \mathsf{alt}(\sigma,\rho)\big) \coloneqq \Big(\mathrm{head}(\sigma),\ \big(\mathsf{alt}(\mathsf{tail}(\rho),\mathsf{alt}(\mathrm{tail}(\tau),\mathrm{tail}(\sigma))),\ \mathsf{alt}(\mathrm{tail}(\rho),\mathrm{tail}(\sigma))\big)\Big)$$

for all $\sigma,\tau,\rho \in A^{\mathbb{N}}$. Then, letting $\pi_1,\pi_2\colon R \to A^{\mathbb{N}}$ be defined by $\pi_1(x,y) \coloneqq x$ and $\pi_2(x,y) \coloneqq y$ for all $(x,y) \in R$, we claim that the diagram

$$\begin{tikzcd}
A \times A^{\mathbb{N}} & A \times R \arrow[l, "\mathrm{id}_A \times \pi_1"'] \arrow[r, "\mathrm{id}_A \times \pi_2"] & A \times A^{\mathbb{N}} \\
A^{\mathbb{N}} \arrow[u, "\langle \mathrm{head}, \mathrm{tail}\rangle"] & R \arrow[u, "e"] \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & A^{\mathbb{N}} \arrow[u, "\langle \mathrm{head}, \mathrm{tail}\rangle"']
\end{tikzcd}$$

in **Sets** commutes, i.e. π_1 and π_2 are both $(A \times (-))$-coalgebra homomorphisms from (R,e) to the final coalgebra $(A^{\mathbb{N}}, \langle \mathrm{head}, \mathrm{tail}\rangle)$. The right-hand square commutes because, as observed before, $\mathrm{tail}(\mathsf{alt}(\sigma,\rho)) = \mathsf{alt}(\mathrm{tail}(\rho),\mathrm{tail}(\sigma))$ for all $\sigma,\rho \in A^{\mathbb{N}}$. Using this observation, we also obtain the equalities

$$\mathsf{alt}(\mathsf{alt}(\mathrm{tail}(\rho),\mathrm{tail}(\tau)),\mathrm{tail}(\sigma)) = \mathsf{alt}(\mathsf{tail}(\mathsf{alt}(\tau,\rho)),\mathrm{tail}(\sigma)) = \mathrm{tail}(\mathsf{alt}(\sigma,\mathsf{alt}(\tau,\rho)))$$

and, similarly, $\mathsf{alt}(\mathsf{tail}(\rho),\mathsf{alt}(\mathrm{tail}(\tau),\mathrm{tail}(\sigma))) = \mathrm{tail}(\mathsf{alt}(\mathsf{alt}(\sigma,\tau),\rho))$, for all $\sigma,\tau,\rho \in A^{\mathbb{N}}$. These imply that the left-hand square in the diagram above commutes. Now, the finality of $(A^{\mathbb{N}}, \langle \mathrm{head}, \mathrm{tail}\rangle)$ in $\mathbf{CoAlg}(F)$ implies $\pi_1 = \pi_2$. Therefore $R \subseteq \{\,(\sigma,\sigma) : \sigma \in A^{\mathbb{N}}\,\}$, giving us our result.

3. Fix any $((-) \times B)^A$-coalgebra $c\colon X \to (X \times B)^A$ and let $X \xleftarrow{\pi_1} X \times B \xrightarrow{\pi_2} B$ be the relevant projections. Any $((-) \times B)^A$-coalgebra homomorphism $f\colon (X,c) \to (Z,\zeta)$ must satisfy $\zeta \circ f = (f \times \mathrm{id}_B)^A \circ c$, or equivalently,

$$\begin{aligned}
&f(x)(\langle a,a,a,\dots\rangle)(0) = \pi_2\big(c(x)(a)\big) \quad \text{and} \\
&f(x)(a\cdot\alpha)(n+1) = f\Big(\pi_1\big(c(x)(a)\big)\Big)(\alpha)(n)
\end{aligned}$$

for all $x \in X$, $a \in A$, $\alpha \in A^{\mathbb{N}}$, and $n \in \mathbb{N}$. In this case, for any $x \in X$ and $\alpha \in A^{\mathbb{N}}$, if we write $\alpha = a \cdot \alpha'$ where $a \in A$ and $\alpha' \in A^{\mathbb{N}}$, then we must have

$$f(x)(\alpha)(0) = f(x)(\langle a, a, a, \dots \rangle)(0) = \pi_2\big(c(x)(a)\big),$$

where the first equality follows from the definition of Z. Furthermore, if, for some $n \in \mathbb{N}$, we already know all the values of $f(y)(\beta)(n)$ for all $y \in X$ and $\beta \in A^{\mathbb{N}}$, then

$$f(x)(\alpha)(n+1) = f(x)(a \cdot \alpha')(n+1) = f\Big(\pi_1\big(c(x)(a)\big)\Big)(\alpha')(n).$$

These uniquely determine the $((-) \times B)^A$-coalgebra homomorphism $f\colon (X, c) \to (Z, \zeta)$. $\square$

Exercise 2.3.3
Assume a category $\mathbb{C}$ with a final object $1 \in \mathbb{C}$. Call a functor $F\colon \mathbb{C} \to \mathbb{C}$ affine if it preserves the final object: the map $F(1) \to 1$ is an isomorphism. Prove that the inverse of this map is the final F-coalgebra, i.e. the final object in the category **CoAlg**(F)*. (Only a few of the functors F that we consider are affine; examples are the identity functor, the non-empty powerset functor, or the distribution functor from Section 4.1. Affine functions occur especially in probabilitistic computation.)*

Solution. Let $1 \underset{\cong}{\xrightarrow{\zeta}} F1$ be an isomorphism. Given any other F-coalgebra $A \xrightarrow{\alpha} FA$, the unique morphism $A \xrightarrow{f} 1$ is a homomorphism of coalgberas: the diagram

$$\begin{tikzcd} FA \arrow[r, "Ff"] & F1 \\ A \arrow[u, "\alpha"] \arrow[r, "f"'] & 1 \arrow[u, "\zeta"', "\cong"] \end{tikzcd}$$

in $\mathbb{C}$ commutes simply because $F1$ is also a final object. $\square$

Exercise 2.3.4
Let Z be the (state space of the) final coalgebra of the binary tree functor $X \mapsto 1 + (A \times X \times X)$. Define by coinduction a mirror function $\mathsf{mir}\colon Z \to Z$ *which (deeply) exchanges the subtrees. Prove, again by coinduction, that* $\mathsf{mir} \circ \mathsf{mir} = \mathrm{id}_Z$*. Can you tell what the elements of Z are?*

Solution. Specifying a coalgebra $f\colon X \to 1 + (A \times X \times X)$ is equivalent to specifying a set $\ker(f) \coloneqq \{\, x \in X : f(x) \in 1 \,\}$, a function $f_A\colon X \setminus \ker(f) \to A$, and two more functions $f_1, f_2, \colon X \setminus \ker(f) \to X$. In this situation, we can form a function $\langle f_A, f_1, f_2 \rangle\colon X \setminus \ker(f) \to A \times X \times X$ mapping each $x \in X \setminus \ker(f)$ to the triple $(f_A(x), f_1(x), f_2(x)) \in A \times X \times X$, so that

$$f(x) = \begin{cases} *, & \text{if } x \in \ker(f), \\ (f_A(x), f_1(x), f_2(x)), & \text{if } x \in X \setminus \ker(f) \end{cases}$$

for all $x \in X$, where $*$ is the unique element of 1.

Fix a set A, and let $Z \underset{\cong}{\xrightarrow{\zeta}} 1 + (A \times Z \times Z)$ be the final coalgebra of the endofunctor $X \mapsto 1 + (A \times X \times X)$ on **Sets**. The coalgebra $Z \xrightarrow{\zeta'} 1 + (A \times Z \times Z)$, defined by $\ker(\zeta') \coloneqq \ker(\zeta)$, $\zeta'_A \coloneqq \zeta_A$, $\zeta'_1 \coloneqq \zeta_2$, and $\zeta'_2 \coloneqq \zeta_1$, admits a unique coalgebra homomorphism $(Z, \zeta') \xrightarrow{\mathsf{mir}} (Z, \zeta)$ by the finality of (Z, ζ). That is, we have the commuting diagram

$$\begin{tikzcd}
1+(A\times Z\times Z) \arrow[r, dashed, "\mathrm{id}_1+(\mathrm{id}_A\times\mathsf{mir}\times\mathsf{mir})"] & 1+(A\times Z\times Z) \\
Z \arrow[u, "\zeta'"] \arrow[r, dashed, "\exists!\mathsf{mir}"'] & Z \arrow[u, "\cong"', "\zeta"]
\end{tikzcd}$$

in **Sets**. Then, given a state $z \in Z$, we have that $z \in \ker(\zeta)$ if and only if $\mathsf{mir}(z) \in \ker(\zeta)$. Furthermore, given a state $z \in Z \setminus \ker(\zeta)$, we have $\big(\zeta_A(z), \zeta_1(\mathsf{mir}(z)), \zeta_2(\mathsf{mir}(z))\big) = \big(\zeta_A(z), \mathsf{mir}(\zeta_2(z)), \mathsf{mir}(\zeta_1(z))\big)$. Therefore, for any binary tree T arising from the coalgebra $Z \xrightarrow{\zeta} 1+(A\times Z\times Z)$ starting from an initial state $z \in Z$, the tree which arises by starting from $\mathsf{mir}(z)$ will be the tree T but with all subtrees deeply exchanged.

Now, it is easy to check that the square

$$\begin{tikzcd}
1+(A\times Z\times Z) \arrow[r, "\mathrm{id}_1+(\mathrm{id}_A\times\mathsf{mir}\times\mathsf{mir})"] & 1+(A\times Z\times Z) \\
Z \arrow[u, "\zeta", "\cong"'] \arrow[r, "\mathsf{mir}"'] & Z \arrow[u, "\zeta'"']
\end{tikzcd}$$

in **Sets** also commutes, making mir also a coalgebra homomorphism from (Z, ζ) to (Z, ζ'). Thus we have a coalgebra homomorphism $(Z,\zeta) \xrightarrow{\mathsf{mir}\circ\mathsf{mir}} (Z,\zeta)$, which must be equal to id_Z by the finality of (Z,ζ).

The set Z is the collection of all (finite and infinite) binary trees. The map ζ sends the empty binary tree to the unique element in 1, and ζ sends a non-empty binary tree $\begin{tikzcd}[column sep=small] & a \arrow[dl] \arrow[dr] & \\ T_1 & & T_2 \end{tikzcd}$, where $a \in A$ and T_1 and T_2 are binary trees, to the triple (a, T_1, T_2). □

Exercise 2.3.5

Recall the decimal representation coalgebra $\mathsf{nextdec}\colon [0,1) \to 1 + (\{0,1,\ldots,9\} \times [0,1))$ *from Example 1.2.2, with its behaviour map* $\mathrm{beh}_{\mathsf{nextdec}}\colon [0,1) \to \{0,1,\ldots,9\}^\infty$*. Prove that this behaviour map is a split mono: there is a map* e *in the reverse direction for which we have* $e \circ \mathrm{beh}_{\mathsf{nextdec}} = \mathrm{id}_{[0,1)}$*. (The behaviour map is not an isomorphism, because both numbers* 5 *and* $49999\cdots$*, considered as sequences in* $\{0,1,\ldots,9\}^\infty$*, represent the fraction* $\frac{1}{2} \in [0,1)$*. See other representations as continued fractions in Pavlović and Pratt (2002) or Niqui (2004) which do yield isomorphisms.)*

Solution. Define $e\colon \{0,\ldots,9\}^\infty \to [0,1)$ by $e(\sigma) := \sum_{n=1}^{\infty} \frac{\sigma_n}{10^n}$ for all infinite sequences $\sigma = (\sigma_0, \sigma_1, \ldots) \in \{0,\ldots,9\}^\infty$, and analogously for finite sequences with finite sums. Following on from the notation of Exercise 1.2.1, we have

$$\begin{aligned}
e(\mathrm{beh}_{\mathsf{nextdec}}(0)) &= e\Big(\mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\perp\}} \ \cup\ (\mathrm{id}_{\{0,\ldots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\big)\big(\mathsf{nextdec}(0)\big)\Big)\Big) \\
&= e\Big(\mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\perp\}} \ \cup\ (\mathrm{id}_{\{0,\ldots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\big)(\perp)\Big)\Big) \\
&= e(\mathsf{next}^{-1}(\perp))
\end{aligned}$$

$$
\begin{aligned}
&= e((0,0,\dots))\\
&= 0.
\end{aligned}
$$

Also, for any $r = \sum_{n=1}^{\infty} \frac{r_n}{10^n} \in (0,1)$, with $r_1, r_2, \dots \in \{0,\dots,9\}$, we have

$$
\begin{aligned}
e(\mathrm{beh}_{\mathsf{nextdec}}(r)) &= e\Big(\mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\perp\}} \;\cup\; (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\big)\big(\mathsf{nextdec}(r)\big)\Big)\Big)\\
&= e\Big(\mathsf{next}^{-1}\Big(\big(\mathrm{id}_{\{\perp\}} \;\cup\; (\mathrm{id}_{\{0,\dots,9\}} \times \mathrm{beh}_{\mathsf{nextdec}})\big)\big((r_1, 10r - r_1)\big)\Big)\Big)\\
&= e\Big(\mathsf{next}^{-1}\big(r_1, \mathrm{beh}_{\mathsf{nextdec}}(10r - r_1)\big)\Big)\\
&= e\big(r_1 \cdot \mathrm{beh}_{\mathsf{nextdec}}(10r - r_1)\big)\\
&= \frac{r_1}{10} + \frac{1}{10} e\left(\mathrm{beh}_{\mathsf{nextdec}}\left(\sum_{n=1}^{\infty} \frac{r_{n+1}}{10^n}\right)\right)\\
&= r.
\end{aligned}
$$

Thus $e \circ \mathrm{beh}_{\mathsf{nextdec}} = \mathrm{id}_{[0,1)}$. □

Exercise 2.3.6

This exercise is based on Jacobs (1996, lemma 5.6).

1. *Fix three sets A, B, C, and consider the simple polynomial functor*

$$X \mapsto (C + (X \times B))^A.$$

Show that its final coalgebra can be described as the set of functions

$$
\begin{aligned}
Z = \{\, \varphi \in (C+B)^{A^+} \mid \forall \sigma \in A^+.\forall c \in C.\varphi(\sigma) = \kappa_1(c) \Rightarrow\\
\forall \tau \in A^\star.\varphi(\sigma \cdot \tau) = \kappa_1(c) \,\}.
\end{aligned}
$$

Once such functions $\varphi \in Z$ hit C, they keep this value in C. Here we write A^+ for the subset of $A^\star$ of non-empty finite sequences. The associated coalgebra structure $\zeta\colon Z \xrightarrow{\cong} (C + (Z \times B))^A$ is given by

$$
\zeta(\varphi)(a) = \begin{cases} \kappa_1(c) & \text{if } \varphi(\langle a \rangle) = \kappa_1(c)\\ \kappa_2(\varphi', b) & \text{if } \varphi(\langle a \rangle) = \kappa_2(b) \text{ where } \varphi'(\sigma) = \sigma(a \cdot \sigma). \end{cases}
$$

2. *Check that the fact that the set B^∞ of both finite and infinite sequences is the final coalgebra of the functor $X \mapsto 1 + (X \times B)$ is a special case of this.*

3. *Generalise the result in (1) to functors of the form*

$$X \mapsto (C_1 + (X \times B_1))^{A_1} \times \dots \times (C_n + (X \times B_n))^{A_n}$$

using this time as state space of the final coalgebra the set

$$
\begin{aligned}
\Big\{ \varphi \in (C+B)^{A^+} \;\Big|\; &\forall \sigma \in A^\star.\forall i \le n.\\
&\forall a \in A_i.\varphi(\sigma \cdot \kappa_i(a)) \in \kappa_1[\kappa_i[C_i]] \vee \varphi(\sigma \cdot \kappa_i(a)) \in \kappa_2[\kappa_i[B_i]]\\
&\wedge \forall c \in C_i.\sigma \neq \langle\rangle \wedge \varphi(\sigma) = \kappa_1(\kappa_i(c))\\
&\Rightarrow \forall \tau \in A^\star.\varphi(\sigma \cdot \tau) = \kappa_1(\kappa_i(c)) \Big\}
\end{aligned}
$$

where $A = A_1 + \dots + A_n$, $B = B_1 + \dots + B_n$ and $C = C_1 + \dots + C_n$.

4. *Show how classes as in (1.10) fit into this last result. Hint: Use that $S + (S \times E) \cong (S \times 1) + (S \times E) \cong S \times (1 + E)$, using distributivity from Exercise 2.1.7.*

Solution. Without loss of generality, we shall assume that all coproducts are simply disjoint unions, so that we may say $X \subseteq X + Y$ and $Y \subseteq X + Y$ for any sets X and Y. This lets us drop all the coprojection morphisms $X \xrightarrow{\kappa_1} X + Y \xleftarrow{\kappa_2} Y$, and (for instance) simply write $x \in X \subseteq X + Y$ whenever we mean $\kappa_1(x) \in X + Y$, for all $x \in X$.

1. Fix an arbitrary $(C + ((-) \times B)^A$-coalgebra $\xi\colon X \to (C + (X \times B))^A$. Any function $h\colon X \to Z$ making the diagram

$$\begin{tikzcd}
(C + (X \times B))^A \arrow[r, "(\mathrm{id}_C + (h \times \mathrm{id}_B))^A"] & (C + (Z \times B))^A \\
X \arrow[u, "\xi"] \arrow[r, "h"'] & Z \arrow[u, "\cong"', "\zeta"]
\end{tikzcd}$$

in **Sets** commute must satisfy all of the following.

   (a) For all $x \in X$ and $a \in A$:
      (i) if $\xi(x)(a) \in C$ then $h(x)(\langle a \rangle) = \xi(x)(a)$;
      (ii) if $\xi(x)(a) \in X \times B$ then, writing $(x', b) \coloneqq \xi(x)(a)$, we have $h(x)(\langle a \rangle) = b$.

   (b) For all $x \in X$, $\sigma \in A^+$, and $a \in A$:
      (i) if $\xi(x)(a) \in C$ then $h(x)(a \cdot \sigma) = h(x)(\langle a \rangle)$;
      (ii) if $\xi(x)(a) \in X \times B$ then, writing $(x', b) \coloneqq \xi(x)(a) \in X \times B$, we have $h(x)(a \cdot \sigma) = h(x')(\sigma)$.

   These requirements inductively define a unique $h(x) \in Z$ for all $x \in X$.

2. Taking $A = C = 1$ in part (1) gives

$$\begin{aligned} Z &\cong \{\, \varphi \in (1 + B)^{\mathbb{N}} \;:\; \text{for all } n \in \mathbb{N}, \text{ if } \varphi(n) \in 1 \text{ then } \varphi(n+1) \in 1 \,\} \\ &\cong B^{\infty}. \end{aligned}$$

3. Fix a natural number $n \geq 1$, fix sets $A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n$, and let $A \coloneqq \sum_{i=1}^{n} A_i$, $B \coloneqq \sum_{i=1}^{n} B_i$, and $C \coloneqq \sum_{i=1}^{n} C_i$. Let $F\colon \mathbf{Sets} \to \mathbf{Sets}$ be the functor defined on objects by $FX \coloneqq (C_1 + (X \times B_1))^{A_1} \times \cdots \times (C_n + (X \times B_n))^{A_n}$ for all sets X.

   Let Z be the set of all functions $\varphi\colon A^+ \to C + B$ satisfying the following property: for $\sigma \in A^*$, $i \in \{1, \ldots, n\}$, and $a \in A_i$, we have $\varphi(\sigma \cdot a) \in B_i + C_i$; furthermore if $\sigma \in A^+$ and $\varphi(\sigma) \in C_i$, then $\varphi(\sigma \cdot a) = \varphi(\sigma)$. Define $\zeta\colon Z \to FZ$ by stipulating that for all $\varphi \in Z$ and $i \in \{1, \ldots, n\}$, the i-th component of the tuple $\zeta(\varphi)$ is the function which maps an element $a \in A_i$ to

$$\begin{cases} \varphi(\langle a \rangle), & \text{if } \varphi(\langle a \rangle) \in C_i, \\ (\varphi', b), & \text{if } \varphi(\langle a \rangle) = b \in B_i \text{ and } \varphi'\colon A^+ \to C + B \text{ is defined by } \varphi'(\sigma) \coloneqq \varphi(a \cdot \sigma) \text{ for all } \sigma \in A^+. \end{cases}$$

   We can prove that (Z, ζ) is the final F-coalgebra similarly as in part (1). The unique F-coalgebra homomorphism h from an arbitrary F-coalgebra (X, ξ) to (Z, ζ) must map each $x \in X$ to the function $h(x) \in Z$ defined as follows.

(a) For all $a \in A$, letting $i \in \{1, \dots, n\}$ be such that $a \in A_i$,

(i) if the i-th projection of $\xi(x)$ maps a to some $c \in C_i$, then we define $h(x)(\langle a \rangle) \coloneqq c$;

(ii) if the i-th projection of $\xi(x)$ maps a to some $(x', b) \in X \times B_i$, then we define $h(x)(\langle a \rangle) \coloneqq b$.

(b) For all $\sigma \in A^+$ and $a \in A$, letting $i \in \{1, \dots, n\}$ be such that $a \in A_i$,

(i) if the i-th component of $\xi(x)$ maps a to some $c \in C_i$, then we define $h(x)(a \cdot \sigma) \coloneqq c$;

(ii) if the i-th component of $\xi(x)$ maps a to some $(x', b) \in X \times B_i$, then we define $h(x)(a \cdot \sigma) \coloneqq h(x')(\sigma)$.

4. The codomain of each method $\mathsf{meth}_i \colon S \to \{\bot\} \cup S \cup (S \times E)$ is isomorphic to $1 + (S \times (1 + E))$, making meth_i a $(1 + ((-) \times (1 + E))$-coalgebra. Each attribute $\mathrm{at}_i \colon S \to D_i$ can also be viewed as the coalgebra of a constant functor; constant functors are subsumed by part (1). Combining all the attributes and methods together gives a coalgebra of a functor as in part (3). □

Exercise 2.3.7

For a topological space A consider the set $A^{\mathbb{N}}$ of streams with the product topology (the least topology that makes the projections $\pi_n \colon A^{\mathbb{N}} \to A$ continuous).

1. *Check that the* head: $A^{\mathbb{N}} \to A$ *and* tail: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ *operations are continuous.*

2. *Prove that the functor $A \times (-) \colon \mathbf{Sp} \to \mathbf{Sp}$ has $A^{\mathbb{N}}$ as the final coalgebra.*

3. *Show that in the special case where A carries the discrete topology (in which every subset is open) the product topology on $A^{\mathbb{N}}$ is given by basic open sets $\uparrow \sigma = \{ \sigma \cdot \tau \mid \tau \in A^{\mathbb{N}} \}$, for $\sigma \in A^\star$ as in Example 2.3.10.*

Solution.

1. The function head: $A^{\mathbb{N}} \to A$ is one of the projections, so it is continuous by definition of the product topology. For an open set $U \subseteq A^{\mathbb{N}}$, the preimage of U under the function tail: $A^{\mathbb{N}} \to A^{\mathbb{N}}$ is $\{ (a, u_1, u_2, u_3, \dots) : a \in A \text{ and } (u_1, u_2, u_3, \dots) \in U \} \cong A \times U$, which is open.

2. The structure map on the desired final coalgebra is the same as in the case when the underlying category is **Sets**. That is, $(A^{\mathbb{N}}, \langle \mathrm{head}, \mathrm{tail} \rangle)$ is the final coalgebra of $A \times (-) \colon \mathbf{Sp} \to \mathbf{Sp}$. To check this, we just need to check that the unique behaviour function beh_c, as obtained in **Sets**, from an $(A \times (-))$-coalgebra (X, c) to $(A^{\mathbb{N}}, \langle \mathrm{head}, \mathrm{tail} \rangle)$ is also continuous.

   The function $\mathrm{beh}_c \colon X \to A^{\mathbb{N}}$ is defined by
$$\mathrm{beh}_c(x) \coloneqq a \cdot \mathrm{beh}_c(x'), \quad \text{where } c(x) = (a, x')$$
   for all $x \in X$. Write $c = \langle c_A, c_X \rangle$. Fix any basic open subset $U = \prod_{n=0}^{\infty} U_n$ of $A^{\mathbb{N}}$, where each U_n is a basic open subset of A, and there exists $N \in \mathbb{N}$ such that $U_n = A$ for all $n > N$. Then
$$\mathrm{beh}_c^{-1}(U) = \{ x \in X : \mathrm{beh}_c(x) \in U \} = \bigcap_{n=0}^{N} \{ x \in X : c_A(c_X^n(x)) \in U_n \} = \bigcap_{n=0}^{N} \big((c_A \circ c_X^n)^{-1}(U) \big),$$
   so $\mathrm{beh}_c^{-1}(U)$ is open in X.

3. If A carries the discrete topology, then the topology of $A^{\mathbb{N}}$ is generated by the sets $\prod_{n=0}^{\infty} U_n$ where each U_n is a subset of A, and there exists $N \in \mathbb{N}$ such that $U_n = A$ for all $n > N$. It is clear that we can rewrite these sets $\prod_{n=0}^{\infty} U_n$ as unions of sets of the form $\sigma \uparrow$ with $\sigma \in A^\star$. □

Exercise 2.3.8

1. *Note that the assignment $A \mapsto A^{\mathbb{N}}$ yields a functor* **Sets** $\to$ **Sets**.
2. *Prove the general result: consider a category $\mathbb{C}$ with a functor $F\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ in two variables. Assume that for each object $A \in \mathbb{C}$, the functor $F(A, -)\colon \mathbb{C} \to \mathbb{C}$ has a final coalgebra $Z_A \xrightarrow{\cong} F(A, Z_A)$. Prove that the mapping $A \mapsto Z_A$ extends to a functor $\mathbb{C} \to \mathbb{C}$.*

Solution.

1. We have already noted this for categories with exponents in our solution to Exercise 2.1.8.3.
2. For $A \in \mathbb{C}$, let $Z_A \underset{\cong}{\xrightarrow{\zeta}} F(A, Z_A)$ denote the final $F(A, -)$-coalgebra. For a morphism $A \xrightarrow{f} B$ in $\mathbb{C}$, we define $Z_A \xrightarrow{Z_f} Z_B$ to be the unique $F(B, -)$-coalgebra homomorphism from $(Z_A, F(f, \mathrm{id}_{Z_A}) \circ \zeta_A)$ to (Z_B, ζ_B).

$$\begin{tikzcd}
F(B, Z_A) \arrow[rr, dashed, "F(\mathrm{id}_B, Z_f)"] & & F(B, Z_B) \\
F(A, Z_A) \arrow[u, "F(f, \mathrm{id}_{Z_A})"] & & \\
Z_A \arrow[u, "\zeta_A", "\cong"'] \arrow[rr, dashed, "\exists! Z_f"'] & & Z_B \arrow[uu, "\cong", "\zeta_B"']
\end{tikzcd}$$

If $A = B$ and $f = \mathrm{id}_A$, then $F(f, \mathrm{id}_{Z_A}) \circ \zeta_A = F(\mathrm{id}_A, \mathrm{id}_{Z_A}) \circ \zeta_A = \mathrm{id}_{F(A, Z_A)} \circ \zeta_A = \zeta_A$, so that we obtain $Z_{\mathrm{id}_A} = \mathrm{id}_{Z_A}$ by finality of (Z_A, ζ_A) in $\mathbf{CoAlg}(F(A, -))$.

If we have two morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{C}$, then we have the commuting diagram

$$\begin{tikzcd}
F(C, Z_A) \arrow[r, "F(\mathrm{id}_B, Z_f)"] & F(C, Z_B) \arrow[r, "F(\mathrm{id}_C, Z_g)"] & F(C, Z_C) \\
F(B, Z_A) \arrow[u, "F(g, \mathrm{id}_{Z_A})"] \arrow[r, "F(\mathrm{id}_B, Z_f)"] & F(B, Z_B) \arrow[u, "F(g, \mathrm{id}_{Z_B})"'] & \\
F(A, Z_A) \arrow[u, "F(f, \mathrm{id}_{Z_A})"] & & \\
Z_A \arrow[u, "\zeta_A", "\cong"'] \arrow[r, "Z_f"'] & Z_B \arrow[uu, "\cong", "\zeta_B"'] \arrow[r, "Z_g"'] & Z_C \arrow[uuu, "\cong", "\zeta_C"']
\end{tikzcd}$$

in $\mathbb{C}$, making $Z_g \circ Z_f$ an $F(C, -)$-coalgebra homomorphism from $(Z_A, F(gf, \mathrm{id}_{Z_A}) \circ \zeta_A)$ to (Z_C, ζ_C). The finality of (Z_C, ζ_C) in $\mathbf{CoAlg}(F(C, -))$ then implies that $Z_g \circ Z_f = Z_{gf}$. $\square$

2.4 Algebras

Exercise 2.4.1

Check that the set $\mathbb{N}\cup\{\infty\}$ of extended natural numbers is the final coalgebra of the functor $X \mapsto 1+X$, as a special case of finality of A^∞ for $X \mapsto 1+(A\times X)$. Use this fact to define appropriate addition and multiplication operations $(\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\}) \to \mathbb{N}\cup\{\infty\}$; see Rutten (2000).

Solution. That $\mathbb{N}\cup\{\infty\}$ is the carrier of the final coalgebra of the functor $1+(-)\colon \mathbf{Sets}\to\mathbf{Sets}$ follows from the fact that $1^\infty \cong \mathbb{N}\cup\{\infty\}$ in **Sets**.

Let $*$ be the unique element in 1. The structure map pred of the final coalgebra of $1+(-)\colon \mathbf{Sets}\to \mathbf{Sets}$ will:

1. map 0 to $*$;
2. for $n\in\mathbb{N}$, map $n+1$ to n;
3. map ∞ to ∞.

Define $c\colon (\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\}) \to 1+((\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\}))$ by

$$c(x,y) \coloneqq \begin{cases} *, & \text{if } x=y=0,\\ (\infty,\infty), & \text{if } x=0 \text{ or } y=\infty,\\ (\mathrm{pred}(x),y), & \text{if } x,y\in\mathbb{N} \text{ and } x>0,\\ (x,\mathrm{pred}(y)), & \text{if } x=0,\ y\in\mathbb{N} \text{ and } y>0, \end{cases}$$

for all $(x,y)\in(\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\})$. Then there is a unique $(1+(-))$-coalgebra add from $((\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\}),c)$ to $(\mathbb{N}\cup\{\infty\},\mathrm{pred})$.

$$\begin{tikzcd} 1+((\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\})) \arrow[r, dashed, "\mathrm{id}_1+\mathrm{add}"] & 1+(\mathbb{N}\cup\{\infty\}) \\ (\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\}) \arrow[u, "c"] \arrow[r, dashed, "\exists!\mathrm{add}"'] & \mathbb{N}\cup\{\infty\} \arrow[u, "\cong"', "\mathrm{pred}"] \end{tikzcd}$$

This function $\mathrm{add}\colon (\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\})\to\mathbb{N}\cup\{\infty\}$ works as our addition operation. Indeed, the equation $\mathrm{pred}(\mathrm{add}(\infty,\infty)) = \mathrm{add}(\infty,\infty)$ from the commuting diagram implies that $\mathrm{add}(\infty,\infty)=\infty$. The other properties we expect from addition on $\mathbb{N}\cup\{\infty\}$ then follow.

Now define $d\colon (\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\}) \to 1+((\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\}))$ by

$$d(x,y) \coloneqq \begin{cases} *, & \text{if } x=0 \text{ or } y=0,\\ (\infty,\infty), & \text{if } x\neq 0\neq y, \text{ and } x=\infty \text{ or } y=\infty,\\ (\mathrm{pred}(x),\mathrm{pred}(x)), & \text{if } x,y\in\mathbb{N} \text{ and } x=y>0, \end{cases}$$

Now define $d\colon (\mathbb{N}\cup\{\infty\})^3 \to 1+((\mathbb{N}\cup\{\infty\})^3$ as follows:

1. if $x=0$, and $y=0$ or $z=0$, then $d(x,y,z)\coloneqq *$;
2. if $x=0$, both $y\neq 0$ and $z\neq 0$, and $y=\infty$ or $z=\infty$, then $d(x,y,z)\coloneqq(\infty,0,0)$;
3. if $x=0$ and $y,z\in\mathbb{N}\setminus\{0\}$, then $d(x,y,z)\coloneqq(\mathrm{add}(\mathrm{pred}(y),\mathrm{pred}(z)),\mathrm{pred}(y),\mathrm{pred}(z))$;

4. if $x = \infty$, then $d(x, y, z) \coloneqq (\infty, 0, 0)$;
5. if $x \in \mathbb{N} \setminus \{0\}$, then $d(x, y, z) \coloneqq (\mathrm{pred}(x), y, z)$.

We obtain a unique $(1 + (-))$-coalgebra homomorphism premult: $((\mathbb{N} \cup \{\infty\})^3, d) \to (\mathbb{N} \cup \{\infty\}, \mathrm{pred})$. This function premult will intuitively satisfy $\mathrm{premult}(x, y, z) = x + yz$ for all $x, y, z \in \mathbb{N} \cup \{\infty\}$. Then we can define the desired multiplication operation mult: $(\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\}) \to \mathbb{N} \cup \{\infty\}$ by $\mathrm{mult}(y, z) \coloneqq \mathrm{premult}(0, y, z)$ for all $(y, z) \in (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$. □

Exercise 2.4.2
Define an appropriate size function $\mathsf{BinTree}(A) \to \mathbb{N}$ *by initiality.*

Solution. Fix a set A. The set $\mathsf{BinTree}(A)$ of finite binary trees with nodes labelled by elements of A is the carrier of the initial algebra of the functor $1 + ((-) \times A \times (-))\colon \mathbf{Sets} \to \mathbf{Sets}$. The structure map of this initial algebra is $[\mathsf{nil}, \mathsf{node}]$, which is defined in Example 2.4.5. Define the $(1 + ((-) \times A \times (-))$-algebra $\alpha\colon 1 + (\mathbb{N} \times A \times \mathbb{N}) \to \mathbb{N}$ by

$$\begin{aligned} \alpha(*) &\coloneqq 0, && \text{where } * \text{ is the unique element in } 1, \\ \alpha(n, a, k) &\coloneqq n + k + 1, && \text{for all } (n, a, k) \in \mathbb{N} \times A \times \mathbb{N}. \end{aligned}$$

Then there is a unique $(1 + ((-) \times A \times (-)))$-algebra homomorphism size: $(\mathsf{BinTree}(A), [\mathsf{nil}, \mathsf{node}]) \to (\mathbb{N}, \alpha)$. Then, for $T \in \mathsf{BinTree}(A)$, the number $\mathrm{size}(T)$ will equal the number of nodes in T. □

Exercise 2.4.3
Consider the obvious functions $\mathsf{even}, \mathsf{odd}\colon \mathbb{N} \to 2 = \{0, 1\}$.

1. *Describe the two algebra structures* $1 + 2 \to 2$ *for the functor* $F(X) = 1 + X$ *that define* even *and* odd *by initiality.*
2. *Define a single algebra* $1 + (2 \times 2) \to 2 \times 2$ *that defines the pair* $\langle \mathsf{even}, \mathsf{odd} \rangle$ *by mutual recursion, that is via* $\mathsf{even}(n + 1) = \mathsf{odd}(n)$.

Solution. The function even sends an even natural number to 1 and an odd natural number to 0. The odd function sends an odd natural number to 1 and an even natural number to 0. We will define these using the initiality of the $(1 + (-))$-algebra $(\mathbb{N}, [\mathsf{0}, \mathsf{S}])$.

1. Define $e, o\colon 1 + 2 \to 2$ by

$$\begin{aligned} e(*) &\coloneqq 1, && \text{where } * \text{ is the unique element in } 1, \\ o(*) &\coloneqq 0, \\ e(0) \coloneqq o(0) &\coloneqq 1, \\ e(1) \coloneqq o(1) &\coloneqq 1. \end{aligned}$$

Then even is the unique $(1 + (-))$-algebra homomorphism from $(\mathbb{N}, [\mathsf{0}, \mathsf{S}])$ to $(2, e)$, whereas odd is the unique $(1 + (-))$-algebra homomorphism from $(\mathbb{N}, [\mathsf{0}, \mathsf{S}])$ to $(2, o)$.

2. Define $\alpha\colon 1 + (2 \times 2) \to 2 \times 2$ by

$$\begin{aligned} \alpha(*) &\coloneqq (1, 0), \\ \alpha(0, 0) &\coloneqq (0, 0), \\ \alpha(1, 0) &\coloneqq (0, 1), \\ \alpha(0, 1) &\coloneqq (1, 0), \\ \alpha(1, 1) &\coloneqq (1, 1). \end{aligned}$$

Then $\langle \mathsf{even}, \mathsf{odd} \rangle$ is the unique $(1 + (-))$-algbera homomorphism from $(\mathbb{N}, [\mathsf{0}, \mathsf{S}])$ to $(2 \times 2, \alpha)$. □

Exercise 2.4.4
Show, dually to Proposition 2.1.5, that finite products $(1, \times)$ in a category $\mathbb{C}$ give rise to finite products in a category $\mathbf{Alg}(F)$ of algebras of a functor $F\colon \mathbb{C} \to \mathbb{C}$.

Solution. We define a functor $F^{\mathrm{op}}\colon \mathbb{C}^{\mathrm{op}} \to \mathbb{C}^{\mathrm{op}}$ by the same data used to define F. An F-algebra will then be precisely an F^{op}-coalgebra, and so $(\mathbf{Alg}(F))^{\mathrm{op}} = \mathbf{CoAlg}(F^{\mathrm{op}})$. If $\mathbb{C}$ has finite products, then $\mathbb{C}^{\mathrm{op}}$ has finite coproducts, and so $\mathbf{CoAlg}(F^{\mathrm{op}})$ has finite coproducts by Proposition 2.1.5. So the category $(\mathbf{Alg}(F))^{\mathrm{op}}$ has finite coproducts, meaning the category $\mathbf{Alg}(F)$ has finite products. $\square$

Exercise 2.4.5
Define addition $+\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, multiplication $\cdot\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and exponentiation $(-)^{(-)}\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by initiality. Hint: Use the correspondence (2.11) and define these operations as functions $\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$. Alternatively, one can use Exercise 2.5.5 from the next section.

Solution. Define $a\colon 1 + \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$\begin{aligned} a(*) &\coloneqq \lambda k \in \mathbb{N}.k, && \text{where } * \text{ is the unique element in } 1, \\ a(f) &\coloneqq \lambda k \in \mathbb{N}.(1 + f(k)), && \text{for all } f \in \mathbb{N}^{\mathbb{N}}. \end{aligned}$$

The unique $(1 + (-))$-algebra homomorphism $\mathrm{add}\colon (\mathbb{N}, [0, \mathsf{S}]) \to (\mathbb{N}^{\mathbb{N}}, a)$ satisfies

$$\mathrm{add}(0)(k) = k \quad \text{and} \quad \mathrm{add}(n + 1)(k) = a(\mathrm{add}(n))(k) = 1 + \mathrm{add}(n)(k)$$

for all $n, k \in \mathbb{N}$. Then $\mathrm{add}(n)(k) = n + k$ for all $n, k \in \mathbb{N}$.

Next, define $m\colon 1 + \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$\begin{aligned} m(*) &\coloneqq \lambda k \in \mathbb{N}.0, \\ m(f) &\coloneqq \lambda k \in \mathbb{N}.(k + f(k)), && \text{for all } f \in \mathbb{N}^{\mathbb{N}}. \end{aligned}$$

The unique $(1 + (-))$-algebra homomorphism $\mathrm{mult}\colon (\mathbb{N}, [0, \mathsf{S}]) \to (\mathbb{N}^{\mathbb{N}}, e)$ satisfies

$$\mathrm{mult}(0)(k) = 0 \quad \text{and} \quad \mathrm{mult}(n + 1)(k) = m(\mathrm{mult}(n))(k) = k + \mathrm{mult}(n)(k)$$

for all $n, k \in \mathbb{N}$. Then $\mathrm{mult}(n)(k) = n \cdot k$ for all $n, k \in \mathbb{N}$.

Finally, define $e\colon 1 + \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$\begin{aligned} e(*) &\coloneqq \lambda k \in \mathbb{N}.1, \\ e(f) &\coloneqq \lambda k \in \mathbb{N}.(k \cdot f(k)), && \text{for all } f \in \mathbb{N}^{\mathbb{N}}. \end{aligned}$$

The unique $(1 + (-))$-algebra homomorphism $\exp\colon (\mathbb{N}, [0, \mathsf{S}]) \to (\mathbb{N}^{\mathbb{N}}, e)$ satisfies

$$\exp(0)(k) = 1 \quad \text{and} \quad \exp(n + 1)(k) = e(\exp(n))(k) = k \cdot \exp(n)(k)$$

for all $n, k \in \mathbb{N}$. Then $\exp(n)(k) = k^n$ for all $n, k \in \mathbb{N}$, with the convention that $0^0 = 1$. $\square$

Exercise 2.4.6
Complete the proof of Proposition 2.4.7.

Solution. Recall that Proposition 2.4.7 asserts that for a category $\mathbb{C}$, an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$, an initial F-algebra $FA \underset{\cong}{\xrightarrow{\alpha}} A$, an object $X \in \mathbb{C}$ such that the product $X \times A$ exists, and a morphism $F(X \times A) \xrightarrow{h} X$, there is a unique morphism $A \xrightarrow{f} X$ such that the diagram

$$\begin{tikzcd}
FA \arrow[r, "F\langle f, \mathrm{id}_A\rangle"] \arrow[d, "\alpha"', "\cong"] & F(X \times A) \arrow[d, "h"] \\
A \arrow[r, "f"'] & X
\end{tikzcd}$$

in $\mathbb{C}$ commutes.

For the proof, we first let $F(X \times A) \xrightarrow{h'} X \times A$ be the morphism $h' := \langle h, \alpha(F\pi_2)\rangle$, where $X \xleftarrow{\pi_1} X \times A \xrightarrow{\pi_2} A$ are the associated projections to the product $X \times A$. So we get a unique F-algebra homomorphism $(A, \alpha) \xrightarrow{k} (X \times A, h')$. We thus obtain the commuting diagram

$$\begin{tikzcd}
FA \arrow[rr, "Fk"] \arrow[dd, "\alpha"', "\cong"] & & F(X \times A) \arrow[dd, "{\langle h, \alpha\circ F\pi_2\rangle}"] \arrow[ddl, "h"'] \arrow[dr, "F\pi_2"] & \\
& & & FA \arrow[dd, "\cong"', "\alpha"] \\
A \arrow[rr, "k"'] & & X \times A \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
& X & & A
\end{tikzcd}$$

in $\mathbb{C}$, from which we see that $\pi_2 k$ is an F-algebra homomorphism from the initial F-algebra (A, α) to itself. So $\pi_2 k = \mathrm{id}_A$. Taking $f := \pi_1 k$ gives the desired commmuting square, since

$$f\alpha = h \circ Fk = h \circ F\langle \pi_1 k, \pi_2 k\rangle = h \circ F\langle f, \mathrm{id}_A\rangle.$$

Given any other morphism $A \xrightarrow{g} X$ satisfying $g\alpha = h \circ F\langle g, \mathrm{id}_A\rangle$, then

$$\begin{aligned}
\langle h, \alpha \circ F\pi_2\rangle \circ F\langle g, \mathrm{id}_A\rangle &= \langle h \circ F\langle g, \mathrm{id}_A\rangle, \alpha \circ F\pi_2 \circ F\langle g, \mathrm{id}_A\rangle\rangle \\
&= \langle g\alpha, \alpha \circ \mathrm{id}_{FA}\rangle \\
&= \langle g\alpha, \alpha\rangle \\
&= \langle g, \mathrm{id}_A\rangle \circ \alpha,
\end{aligned}$$

and so $\langle g, \mathrm{id}_A\rangle$ is an F-algebra homomorphism from (A, α) to $(X \times A, h')$. So we get $\langle g, \mathrm{id}_A\rangle = k$ by the initiality of (A, α) in $\mathbf{Alg}(F)$, and thus $g = \pi_1 k = f$. □

Exercise 2.4.7 ('Rolling lemma')
Assume two endofunctors $F, G\colon \mathbb{C} \to \mathbb{C}$ on the same category. Let the composite $FG\colon \mathbb{C} \to \mathbb{C}$ have an initial algebra $\alpha\colon FG(A) \xrightarrow{\cong} A$.

1. *Prove that also the functor $GF\colon \mathbb{C} \to \mathbb{C}$ has an initial algebra, with $G(A)$ as the carrier.*

2. *Formulate and prove a dual result, for final coalgebras.*

Solution.

1. For any GF-algebra $GFX \xrightarrow{\beta} X$, there is a unique FG-algebra homomorphism $(A, \alpha) \xrightarrow{h} (FX, F\beta)$. Then $(GA, G\alpha) \xrightarrow{\beta \circ Gh} (X, \beta)$ is a GF-algebra homomorphism, due to the commutativity of the diagram

$$\begin{tikzcd} GFGA \arrow[r, "GFGh"] \arrow[d, "G\alpha"', "\cong"] & GFGFX \arrow[r, "G\beta"] \arrow[d, "G\beta"] & GFX \arrow[d, "\beta"] \\ GA \arrow[r, "Gh"'] & GFX \arrow[r, "\beta"'] & X \end{tikzcd}$$

in $\mathbb{C}$. Given any other GF-algebra homomorphism $(GA, G\alpha) \xrightarrow{k} (X, \beta)$, the diagram

$$\begin{tikzcd} FGA \arrow[r, "FG(Fk \circ \alpha^{-1})"] \arrow[d, "\alpha"', "\cong"] & FGFX \arrow[d, "F\beta"] \\ A \arrow[r, "Fk \circ \alpha^{-1}"'] & FX \end{tikzcd}$$

in $\mathbb{C}$ commutes, making $(A, \alpha) \xrightarrow{Fk \circ \alpha^{-1}} (FX, F\beta)$ an FG-algebra homomorphism. Indeed,

$$F\beta \circ FG(Fk \circ \alpha^{-1}) = F(\beta \circ GFk \circ G\alpha^{-1}) = Fk = (Fk \circ \alpha^{-1}) \circ \alpha,$$

where the middle equality follows from $(GA, G\alpha) \xrightarrow{k} (X, \beta)$ being a GF-homomorphism. As (A, α) is the initial FG-algebra, we obtain $Fk \circ \alpha^{-1} = h$. Therefore

$$k = \beta \circ GFk \circ G\alpha^{-1} = \beta \circ G(Fk \circ \alpha^{-1}) = \beta \circ Gh.$$

Therefore $(GA, G\alpha)$ is the initial GF-algebra.

2. The dual statement says the following: if we are given two endofunctors $F, G \colon \mathbb{C} \to \mathbb{C}$ on the same category such that there is a terminal FG-coalgebra, then there is also a terminal GF-coalgebra.

   For the dual proof, suppose we are given a terminal FG-coalgebra $A \underset{\cong}{\xrightarrow{c}} FGA$ and we are given an arbitrary GF-coalgebra $X \xrightarrow{d} GFX$. Then there is a unique FG-coalgebra homomorphism $(FX, Fd) \xrightarrow{h} (A, c)$, from which we obtain a GF-coalgebra homomorphism $(X, d) \xrightarrow{Gh \circ d} (GA, Gc)$. Any other GF-coalgebra homomorphism $(X, d) \xrightarrow{k} (GA, Gc)$ will yield an FG-coalgebra homomorphism $(FX, Fd) \xrightarrow{c^{-1} \circ Fk} (A, c)$, from which it follows that $c^{-1} \circ Fk = h$ and hence $k = Gh \circ d$. $\square$

Exercise 2.4.8

This exercise illustrates the analogy between the set $A^\star$ of finite sequences in **Sets** *and the vector space $A^\S$ in* **Vect**, *following Exercise 2.2.12. Recall from Example 2.4.6 that the set $A^\star$ of finite lists of elements of A is the initial algebra of the functor $X \mapsto 1 + (A \times X)$.*

1. *For an arbitrary vector space A, this same functor $1 + (A \times \mathrm{id})$, but considered as an endofunctor on* **Vect**, *can be rewritten as*

$$\begin{aligned} 1 + (A \times X) &\cong A \times X && \text{because } 1 \in \textbf{Vect} \text{ is the initial object} \\ &\cong A + X && \text{because } \times \text{ and } + \text{ are the same in } \textbf{Vect}. \end{aligned}$$

   Prove that the initial algebra of this functor $A + \mathrm{id} \colon \textbf{Vect} \to \textbf{Vect}$ is the vector space $A^\S$ with insertion and shift maps $\mathsf{in} \colon A \to A^\S$ and $\mathsf{sh} \colon A^\S \to A^\S$ defined in Exercise 2.2.12.3.

2. *Check that the behaviour formula from Exercise 2.2.12.4 for a system $A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B$ is obtained as $H \circ \mathrm{int}_{[F,G]} \colon A^{\S} \to B$ using initiality.*

3. *Show that the assignment $A \mapsto A^{\S}$ yields a functor* **Vect** $\to$ **Vect**. *Hint: Actually, this is a special case of the dual of Exercise 2.3.8.*

Solution.

1. Fix an arbitrary $(A + (-))$-algebra $[\alpha, \beta] \colon A + V \to V$. Any linear map $f \colon A^{\S} \to V$ making the diagram

$$\begin{tikzcd} A + A^{\S} \arrow[r, "\mathrm{id}_A + f"] \arrow[d, "{[\mathsf{in},\mathsf{sh}]}"', "\cong"] & A + V \arrow[d, "{[\alpha,\beta]}"] \\ A^{\S} \arrow[r, "f"'] & V \end{tikzcd}$$

in **Vect** commute will have to satisfy both of the following:

   (a) we must have $f(0_A, 0_A, 0_A, \dots) = 0_V$, where 0_A and 0_V are the zero vectors in A and V respectively, since f is a linear map;

   (b) for $v = (v_0, v_1, v_2, \dots) \in A^{\S}$ and $a \in A$, we must also have

$$\begin{aligned} f(a, v_0, v_1, v_2, \dots) &= f(\mathsf{in}(a) +_{A^{\S}} \mathsf{sh}(v)) \\ &= f(\mathsf{in}(a)) +_V f(\mathsf{sh}(v)) \\ &= \alpha(a) +_V \beta(f(v)), \end{aligned}$$

   where $+_{A^{\S}}$ and $+_V$ are the vector addition operations on $A^{\S}$ and V respectively, with the former defined in our solution to Exercise 2.2.12.3.

As any tuple in $v = (v_0, v_1, v_2, \dots) \in A^{\S}$ must be such that $v_n = 0_A$ for all but finitely many $n \in \mathbb{N}$, the requirements above define the unique $(A+(-))$-algebra homomorphism $\mathrm{int}_{[\alpha,\beta]} \colon (A^{\S}, [\mathsf{in}, \mathsf{sh}]) \to (V, [\alpha, \beta])$.

2. Suppose we are given three linear maps $A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B$. By the previous part, there is a unique linear map $\mathrm{int}_{[F,G]} \colon A^{\S} \to X$ such that the diagram

$$\begin{tikzcd} A + A^{\S} \arrow[r, "\mathrm{id}_A + \mathrm{int}_{[F,G]}"] \arrow[d, "{[\mathsf{in},\mathsf{sh}]}"', "\cong"] & A + X \arrow[d, "{[F,G]}"] & \\ A^{\S} \arrow[r, "\mathrm{int}_{[F,G]}"'] & X \arrow[r, "H"'] & B \end{tikzcd}$$

in **Vect** commutes. Then, for $a = (a_0, a_1, \dots, a_n, 0_A, 0_A, \dots) \in A^{\S}$, we have

$$\begin{aligned} &(H \circ \mathrm{int}_{[F,G]})(a_0, a_1, \dots, a_n, 0_A, 0_A, 0_A, \dots) \\ &= ([HF, HG] \circ (\mathrm{id}_A + \mathrm{int}_{[F,G]}) \circ [\mathsf{in}, \mathsf{sh}]^{-1})(a_0, a_1, \dots, a_n, 0_A, 0_A, 0_A, \dots) \\ &= ([HF, HG] \circ (\mathrm{id}_A + \mathrm{int}_{[F,G]}))(a_0, (a_1, \dots, a_n, 0_A, 0_A, 0_A, \dots)) \end{aligned}$$

$$
\begin{aligned}
&= ([HF, HG])(a_0, \mathrm{int}_{[F,G]}(a_1, \ldots, a_n, 0_A, 0_A, 0_A, \ldots)) \\
&= (HF)(a_0) +_B (HG)(\mathrm{int}_{[F,G]}(a_1, \ldots, a_n, 0_A, 0_A, 0_A, \ldots)) \\
&\ \ \vdots \\
&= (HF)(a_0) +_B (HGF)(a_1) +_B \cdots +_B (HG^n F)(a_n)
\end{aligned}
$$

where $+_B$ is the vector addition in B. This agrees with the formula claimed in Exercise 2.2.12.4.

3. Define $F\colon \mathbf{Vect} \times \mathbf{Vect} \to \mathbf{Vect}$ on objects by $F(A, B) \coloneqq A + B$ for vector spaces A and B, so that $F(A, -)$ is the functor we considered in part (1) which has initial algebra $A^{\S}$. The assignment $A \mapsto A^{\S}$ for $A \in \mathbf{Vect}$ then extends to a functor due to Exercise 2.3.8.2.

□

Exercise 2.4.9

Use Exercise 2.1.10 to show that there is a commuting diagram of isomorphisms

$$
\begin{tikzcd}
\mathcal{P}(A^\star)^A \times 2 \arrow[rr, "\cong"] & & \mathcal{P}(1 + (A \times A^\star)) \arrow[dl, "\cong", "\mathcal{P}([\mathsf{nil},\mathsf{cons}])"'] \\
& \mathcal{P}(A^\star) \arrow[ul, "\cong"', "\langle D,E\rangle"] &
\end{tikzcd}
$$

where the coalgebra $\langle D, E\rangle$ is the final Brzozowski deterministic automaton structure from Corollary 2.3.6.2, and $[\mathsf{nil}, \mathsf{cons}]$ is the initial list algebra (from Example 2.4.6).

Solution. Write $2 = \{0, 1\}$ and assume, without loss of generality, that all coproducts are disjoint unions. There are obvious isomorphisms

$$
\begin{tikzcd}
(\mathcal{P}(A^\star))^A \times 2 \arrow[r, "\cong"] & (\mathcal{P}(A^\star))^A \times \mathcal{P}(1) \arrow[r, "\cong"] & \left(\prod_{a\in A} \mathcal{P}(A^\star)\right) \times \mathcal{P}(1) \arrow[dll, "\cong"'] \\
\mathcal{P}\left(\left(\sum_{a\in A} A^\star\right) + 1\right) \arrow[r, "\cong"] & \mathcal{P}((A \times A^\star) + 1) \arrow[r, "\cong"] & \mathcal{P}(1 + (A \times A^\star)),
\end{tikzcd}
$$

where the diagonal isomorphism comes from (an appropriate generalisation of) Exercise 2.1.10, whose composite $(\mathcal{P}(A^\star))^A \times 2 \xrightarrow[\cong]{i} \mathcal{P}(1 + (A \times A^\star))$ is defined by

$$
(\mathcal{P}(A^\star))^A \times 2 \ni (f, t) \overset{i}{\mapsto} \begin{cases} \{\, (a, \sigma) \in A \times A^\star : \sigma \in f(a) \,\}, & \text{if } t = 0, \\ 1 + \{\, (a, \sigma) \in A \times A^\star : \sigma \in f(a) \,\}, & \text{if } t = 1. \end{cases}
$$

Then, for $(f, t) \in (\mathcal{P}(A^\star))^A \times 2$, we have

$$
\begin{aligned}
&(\langle D, E\rangle \circ \mathcal{P}([\mathsf{nil}, \mathsf{cons}]) \circ i)(f, t) \\
&= \begin{cases} (\langle D, E\rangle \circ \mathcal{P}([\mathsf{nil}, \mathsf{cons}]))(\{\, (a, \sigma) \in A \times A^\star : \sigma \in f(a) \,\}), & \text{if } t = 0, \\ (\langle D, E\rangle \circ \mathcal{P}([\mathsf{nil}, \mathsf{cons}]))(1 + \{\, (a, \sigma) \in A \times A^\star : \sigma \in f(a) \,\}), & \text{if } t = 1, \end{cases} \\
&= \begin{cases} \langle D, E\rangle(\{\, a \cdot \sigma : (a, \sigma) \in A \times A^\star \text{ and } \sigma \in f(a) \,\}), & \text{if } t = 0, \\ \langle D, E\rangle(\{\langle\rangle\} \cup \{\, a \cdot \sigma : (a, \sigma) \in A \times A^\star \text{ and } \sigma \in f(a) \,\}), & \text{if } t = 1, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{cases} (\lambda b \in A.\{\, \tau \in A^\star : b \cdot \tau \in \{\, a \cdot \sigma : (a,\sigma) \in A \times A^\star \text{ and } \sigma \in f(a) \,\}\,\},\ 0), & \text{if } t = 0, \\ (\lambda b \in A.\{\, \tau \in A^\star : b \cdot \tau \in \{\langle\rangle\} \cup \{\, a \cdot \sigma : (a,\sigma) \in A \times A^\star \text{ and } \sigma \in f(a) \,\}\,\},\ 1), & \text{if } t = 1, \end{cases} \\
&= (f,t).
\end{aligned}
$$

So $\langle D, E\rangle \circ \mathcal{P}([\mathsf{nil}, \mathsf{cons}]) \circ i = \mathrm{id}_{(\mathcal{P}(A^\star))^A \times 2}$.

Also, for any set $S \subseteq 1 + (A \times A^\star)$, we have

$$
\begin{aligned}
&(i \circ \langle D, E\rangle \circ \mathcal{P}([\mathsf{nil}, \mathsf{cons}]))(S) \\
&= \begin{cases} (i \circ \langle D, E\rangle)(\{\, a \cdot \sigma : (a,\sigma) \in (A \times A^\star) \cap S \,\}), & \text{if } * \notin S, \\ (i \circ \langle D, E\rangle)(\{\langle\rangle\} \cup \{\, a \cdot \sigma : (a,\sigma) \in (A \times A^\star) \cap S \,\}), & \text{if } * \in S, \end{cases} \\
&= \begin{cases} i(\lambda b \in A.\{\, \tau \in A^\star : b \cdot \tau \in \{\, a \cdot \sigma : (a,\sigma) \in (A \times A^\star) \cap S \,\}\,\},\ 0), & \text{if } * \notin S, \\ i(\lambda b \in A.\{\, \tau \in A^\star : b \cdot \tau \in \{\langle\rangle\} \cup \{\, a \cdot \sigma : (a,\sigma) \in (A \times A^\star) \cap S \,\}\,\},\ 1), & \text{if } * \in S, \end{cases} \\
&= \begin{cases} \{\, (b,\rho) \in A \times A^\star : \rho \in \{\, \tau \in A^\star : b \cdot \tau \in \{\, a \cdot \sigma : (a,\sigma) \in (A \times A^\star) \cap S \,\}\,\}\,\}, & \text{if } * \notin S, \\ 1 + \{\, (b,\rho) \in A \times A^\star : \rho \in \{\, \tau \in A^\star : b \cdot \tau \in \{\langle\rangle\} \cup \{\, a \cdot \sigma : (a,\sigma) \in (A \times A^\star) \cap S \,\}\,\}\,\}, & \text{if } * \in S, \end{cases} \\
&= S,
\end{aligned}
$$

where $*$ is the unique element in 1. So $i \circ \langle D, E\rangle \circ \mathcal{P}([\mathsf{nil}, \mathsf{cons}]) = \mathrm{id}_{\mathcal{P}(1+(A \times A^\star))}$.

Therefore $\langle D, E\rangle \circ \mathcal{P}([\mathsf{nil}, \mathsf{cons}])$ has inverse i, giving the commutative diagram claimed. □

Exercise 2.4.10

Suppose we have a 'binary method', say of the form $m\colon X \times X \times A \to 1 + (X \times B)$. There is a well-known trick from Freyd (1990) to use a functorial description also in such cases, namely by separating positive and negative occurences. This leads to a functor of the form $F\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$, which in this case would be $(Y, X) \mapsto (1 + (X \times B))^{Y \times A}$.

In general, for a functor $F\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ define an F-coalgebra to be a map of the form $c\colon X \to F(X, X)$. A homomorphism from $c\colon X \to F(X,X)$ to $d\colon Y \to F(Y,Y)$ is then a map $f\colon X \to Y$ in $\mathbb{C}$ making the following pentagon commute:

$$
\begin{tikzcd}
& F(X,Y) & \\
F(X,X) \arrow[ur, "F(\mathrm{id}_X, f)"] & & F(Y,Y) \arrow[ul, "F(f, \mathrm{id}_Y)"'] \\
X \arrow[u, "c"] \arrow[rr, "f"'] & & Y \arrow[u, "d"']
\end{tikzcd}
$$

1. *Elaborate what this means for the above example $(Y, X) \mapsto (1 + (X \times B))^{Y \times A}$.*

2. *Prove that these coalgebras and their homomorphisms form a category.*

(Such generalised coalgebras are studied systematically in Tews (2001).)

Solution.

1. Suppose the category $\mathbb{C}$ has products and exponents. We first need to understand the functor defined on objects by $(Y, X) \mapsto X^Y$ for all $X, Y \in \mathbb{C}$. This can be made into a functor of type $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ as follows. Given morphisms $A \xrightarrow{f} X$ and $B \xleftarrow{g} Y$ in $\mathbb{C}$, we define the morphism

$A^B \xrightarrow{f^g} X^Y$ in $\mathbb{C}$ as follows: using notation established in our solution to Exercise 2.1.8.3, we have the morphisms $A^B \times Y \xrightarrow{f^B \times g} X^B \times B \xrightarrow{\mathrm{ev}_X^B} X$ in $\mathbb{C}$, so we define $A^B \xrightarrow{f^g} X^Y$ to be the unique morphism in $\mathbb{C}$ such that the diagram

$$\begin{tikzcd}
X^Y \times Y \arrow[rr, "\mathrm{ev}_X^Y"] & & X \\
& X^B \times B \arrow[ur, "\mathrm{ev}_X^B"'] & \\
A^B \times Y \arrow[uu, "f^g \times \mathrm{id}_Y"] \arrow[ur, "f^B \times g"'] & &
\end{tikzcd}$$

in $\mathbb{C}$ commutes. That is, $f^g \coloneqq \Lambda_X^Y\big(\mathrm{ev}_X^B \circ (f^B \times g)\big) = \Lambda_X^Y\Big(\mathrm{ev}_X^B \circ \big(\Lambda_X^B(f \circ \mathrm{ev}_A^B) \times g\big)\Big)$.

Now fix $A, B \in \mathbb{C}$ and define a functor $F\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ on objects by $F(Y, X) \coloneqq (1 + (X \times B))^{Y \times A}$ for $X, Y \in \mathbb{C}$. For two F-coalgebras ($X \xrightarrow{c} (1+(X \times B))^{X \times A}$ and $Y \xrightarrow{d} (1+(Y \times B))^{Y \times A}$, a homomorphism f from (X, c) to (Y, d) is then a morphism $X \xrightarrow{f} Y$ in $\mathbb{C}$ making the diagram

$$\begin{tikzcd}
& (1 + (Y \times B))^{X \times A} & \\
(1 + (X \times B))^{X \times A} \arrow[ur, "(\mathrm{id}_1 + (f \times \mathrm{id}_B))^{\mathrm{id}_X \times \mathrm{id}_A}"] & & (1 + (Y \times B))^{Y \times A} \arrow[ul, "(\mathrm{id}_1 + (\mathrm{id}_Y \times \mathrm{id}_B))^{f \times \mathrm{id}_A}"'] \\
X \arrow[u, "c"] \arrow[rr, "f"'] & & Y \arrow[u, "d"']
\end{tikzcd}$$

in $\mathbb{C}$ commute. The commutativity of the diagram above is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
1 + (X \times B) \arrow[r, "\mathrm{id}_1 + (f \times \mathrm{id}_B)"] & 1 + (Y \times B) \\
(1 + (X \times B))^{X \times A} \times (X \times A) \arrow[u, "\mathrm{ev}_{1+(X \times B)}^{X \times A}"] & (1 + (Y \times B))^{Y \times A} \times (Y \times A) \arrow[u, "\mathrm{ev}_{1+(Y \times B)}^{Y \times A}"'] \\
X \times (X \times A) \arrow[u, "c \times \mathrm{id}_{X \times A}"] \arrow[r, "f \times (f \times \mathrm{id}_A)"'] & Y \times (Y \times A) \arrow[u, "d \times \mathrm{id}_{Y \times A}"']
\end{tikzcd}$$

in $\mathbb{C}$.

2. Now suppose $F\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C}$ is an arbitrary functor on an arbitrary category $\mathbb{C}$. We will let composition in the category of coalgebras simply be composition in $\mathbb{C}$. Given an F-coalgebra $X \xrightarrow{c} F(X, X)$, the diagram

$$\begin{tikzcd}
& F(X, X) & \\
F(X, X) \arrow[ur, "F(\mathrm{id}_X, \mathrm{id}_X)"] & & F(X, X) \arrow[ul, "F(\mathrm{id}_X, \mathrm{id}_X)"'] \\
X \arrow[u, "c"] \arrow[rr, "\mathrm{id}_X"'] & & X \arrow[u, "c"']
\end{tikzcd}$$

in $\mathbb{C}$ clearly commutes, so that id_X is a homomorphism from (X, c). Now, given two homomorphisms of F-coalgebras $(X, c) \xrightarrow{f} (Y, d) \xrightarrow{g} (Z, e)$, we get the commuting diagram

$$\begin{tikzcd}
 & & & F(X,Z) & & & \\
 & & F(X,Y) \arrow[ur, "F(\mathrm{id}_X{,}g)"] & & F(Y,Z) \arrow[ul, "F(f{,}\mathrm{id}_Z)"'] & & \\
 & F(X,X) \arrow[ur, "F(\mathrm{id}_X{,}f)"] & & F(Y,Y) \arrow[ul, "F(f{,}\mathrm{id}_Y)"'] \arrow[ur, "F(\mathrm{id}_Y{,}g)"] & & F(Z,Z) \arrow[ul, "F(g{,}\mathrm{id}_Z)"'] & \\
 & X \arrow[u, "c"] \arrow[rr, "f"'] & & Y \arrow[u, "d"'] \arrow[rr, "g"'] & & Z \arrow[u, "e"'] &
\end{tikzcd}$$

in $\mathbb{C}$, whose outer diagram gives us the desired commuting pentagon

$$\begin{tikzcd}
 & F(X,Z) & \\
F(X,X) \arrow[ur, "F(\mathrm{id}_X{,}gf)"] & & F(Z,Z) \arrow[ul, "F(gf{,}\mathrm{id}_Z)"'] \\
X \arrow[u, "c"] \arrow[rr, "gf"'] & & Z \arrow[u, "e"']
\end{tikzcd}$$

in $\mathbb{C}$ (recalling that F is contravariant in the first argument), so that gf is also a homomorphism from (X, c) to (Z, e). The associtivity and identity laws for the category of these F-coalgebras then follows from the associativity and identity laws for $\mathbb{C}$. $\square$

2.5 Adjunctions, Cofree Coalgebras, Behaviour-Realisation

Exercise 2.5.1

Recall from Exercise 1.4.4 that the assignment $A \mapsto A^\star$ gives a functor $(-)^\star\colon \mathbf{Sets} \to \mathbf{Mon}$ from sets to monoids. Show that this functor is left adjoint to the forgetful functor $\mathbf{Mon} \to \mathbf{Sets}$ which maps a monoid $(M, +, 0)$ to its underlying set M and forgets the monoid structure.

Solution. Fix a set A, a monoid $(M, +, 0)$, and a function $f\colon A \to M$ in **Sets**. Let $\eta_A\colon A \to A^\star$ be the set function defined by $\eta_A(a) \coloneqq \langle a \rangle$ for all $a \in A$. By Exercise 1.4.4, there exists a unique monoid homomorphism $\varphi_{A,(M,+,0)}(f)\colon (A^\star, \cdot, \langle\rangle) \to (M, +, 0)$ such that $\varphi_{A,(M,+,0)}(f) \circ \eta_A = f$. So the assignment

$$\hom_{\mathbf{Sets}}(A, M) \ni f \longmapsto \varphi_{A,(M,+,0)}(f) \in \hom_{\mathbf{Mon}}((A^\star, \cdot, \langle\rangle), (M, +, 0))$$

is a bijection.

We now just need to check that the bijection above is natural in both A and $(M, +, 0)$. Suppose we are given a set function $g\colon B \to A$ and a monoid homomorphism $h\colon (M, +, 0) \to (N, *, 1)$. We claim that the diagram

$$\begin{tikzcd}
\hom_{\mathbf{Sets}}(A, M) \arrow[r, "\varphi_{A,(M,+,0)}", "\cong"'] \arrow[d, "h\circ(-)\circ g"'] & \hom_{\mathbf{Mon}}((A^\star, \cdot, \langle\rangle), (M, +, 0)) \arrow[d, "h\circ(-)\circ g^\star"] \\
\hom_{\mathbf{Sets}}(B, N) \arrow[r, "\cong", "\varphi_{B,(N,*,1)}"'] & \hom_{\mathbf{Mon}}((B^\star, \cdot, \langle\rangle), (N, *, 1))
\end{tikzcd}$$

in **Sets** commutes. Indeed, for any $f \in \hom_{\mathbf{Sets}}(A, M)$ and $b \in B$,

$$\begin{aligned}
(h \circ \varphi_{A,(M,+,0)}(f) \circ g^\star \circ \eta_B)(b) &= (h \circ \varphi_{A,(M,+,0)}(f) \circ g^\star)(\langle b \rangle) \\
&= (h \circ \varphi_{A,(M,+,0)}(f))(\langle g(b) \rangle) \\
&= (h \circ \varphi_{A,(M,+,0)}(f) \circ \eta_A)(g(b)) \\
&= (h \circ f)(g(b)) \\
&= (hfg)(b) \\
&= (\varphi_{B,(N,*,1)}(hfg) \circ \eta_B)(b).
\end{aligned}$$

Therefore $h \circ \varphi_{A,(M,+,0)}(f) \circ g^\star = \varphi_{B,(N,*,1)}(hfg)$. □

Exercise 2.5.2

Check that the bijective correspondences

$$\begin{array}{c}
X \to \mathcal{P}(Y) \\ \hline\hline
\bullet \subseteq X \times Y \\ \hline\hline
\bullet \subseteq Y \times X \\ \hline\hline
Y \to \mathcal{P}(X)
\end{array}$$

induced by the correspondence (2.16) give rise to an adjunction of the form $\mathbf{Sets}^{\mathrm{op}} \leftrightarrows \mathbf{Sets}$.

Solution. Explicitly, the claim is that the contravariant power set functor $\mathcal{P}\colon \mathbf{Sets} \to \mathbf{Sets}^{\mathrm{op}}$ is left adjoint to the contravariant power set functor $\mathcal{P}\colon \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ defined by exactly the same data. For sets X and Y, we have already obtained a bijection $\varphi_{X,Y}\, \hom_{\mathbf{Sets}}(X, \mathcal{P}Y) \to \hom_{\mathbf{Sets}}(Y, \mathcal{P}X) =$

$\hom_{\mathbf{Sets}^{\mathrm{op}}}(\mathcal{P}X, Y)$, by correspondences above. This $\varphi_{X,Y}$ sends a function $h\colon X \to \mathcal{P}Y$ to the function $\lambda y \in Y.\{\, x \in X : y \in h(x)\,\}$. So we just need to check that this bijection $\varphi_{X,Y}$ is natural in both X and Y.

Fix a morphism $X' \xrightarrow{f} X$ in **Sets** and a morphism $Y \xrightarrow{g} Y'$ in $\mathbf{Sets}^{\mathrm{op}}$ (so that g is actually a function from Y' to Y in **Sets**). We wish to show that the diagram

$$\begin{tikzcd}
\hom_{\mathbf{Sets}}(X, \mathcal{P}Y) \arrow[r, "\varphi_{X,Y}", "\cong"'] \arrow[d, "(\mathcal{P}g)\circ(-)\circ f"'] & \hom_{\mathbf{Sets}^{\mathrm{op}}}(\mathcal{P}X, Y) = \hom_{\mathbf{Sets}}(Y, \mathcal{P}X) \arrow[d, "(\mathcal{P}f)\circ(-)\circ g"] \\
\hom_{\mathbf{Sets}}(X', \mathcal{P}(Y')) \arrow[r, "\cong", "\varphi_{X',Y'}"'] & \hom_{\mathbf{Sets}^{\mathrm{op}}}(\mathcal{P}(X'), Y') = \hom_{\mathbf{Sets}}(Y', \mathcal{P}(X'))
\end{tikzcd}$$

in **Sets** commutes, where all indicated instances of composition is composition in **Sets**. The commutativity of this diagram is established as follows: for $h \in \hom_{\mathbf{Sets}}(X, \mathcal{P}Y)$, we have

$$\begin{aligned}
\mathcal{P}f \circ \varphi_{X,Y}(h) \circ g &= \mathcal{P}f \circ \big(\lambda y \in Y.\{\, x \in X : y \in h(x)\,\}\big) \circ g \\
&= \mathcal{P}f \circ \big(\lambda y' \in Y'.\{\, x \in X : g(y') \in h(x)\,\}\big) \\
&= \lambda y' \in Y'.\{\, x' \in X' : g(y') \in h(f(x))\,\} \\
&= \lambda y' \in Y'.\{\, x' \in X' : y' \in (\mathcal{P}g)(h(f(x)))\,\} \\
&= \varphi_{X',Y'}(\mathcal{P}g \circ h \circ f).
\end{aligned}$$

□

Exercise 2.5.3

The intention of this exercise is to show that the mapping $\# \mapsto F_\#$ from arities to functors is functorial. This requires some new notation and terminology. Let $\mathrm{Endo}(\mathbf{Sets})$ *be the category with endofunctors* $F\colon \mathbf{Sets} \to \mathbf{Sets}$ *as objects and natural transformations between them as morphisms. One writes* $\mathbf{Sets}/\mathbb{N}$ *for the slice category over* $\mathbb{N}$*; see Exercise 1.4.3.*

Prove that mapping $\# \mapsto F_\#$ from (2.18) yields a functor $\mathbf{Sets}/\mathbb{N} \to \mathrm{Endo}(\mathbf{Sets})$. *(This functor* $\mathbf{Sets}/\mathbb{N} \to \mathrm{Endo}(\mathbf{Sets})$ *restricts to a 'full and faithful' functor via a suitable restriction of the category* $\mathrm{Endo}(\mathbf{Sets})$*; see Abbott, Altenkirch, Ghani, and McBridge (2003b, theorem 3.4). This means that morphisms $F_{\#_1} \to F_{\#_2}$ in this restricted category are in one-to-one correspondence with morphisms of arities $\#_1 \to \#_2$.)*

Solution. Recall that an arity is a set I together with a function $\#\colon I \to \mathbb{N}$, and that the (simple polynomial) functor $F_\#\colon \mathbf{Sets} \to \mathbf{Sets}$ associated with an arity $\#\colon I \to \mathbb{N}$ is defined on objects by

$$F_\# X \coloneqq \coprod_{i \in I} X^{\#(i)} = \Big\{ (i, x) : i \in I \text{ and } x \in X^{\#(i)} \Big\}$$

for all sets X. Given a set function $f\colon X \to Y$, we define $F_\# f\colon F_\# X \to F_\# Y$ by

$$\begin{aligned}
&(F_\# f)(i, x) \coloneqq \big(i,\ (f(x_1), \dots, f(x_{\#(i)})\big) && \text{for all } i \in I \text{ with } \#(i) \neq 0, \text{ and } x = (x_1, \dots, x_{\#(i)}) \in X^{\#(i)}, \\
&(F_\# f)(i, *) \coloneqq \big(i, *\big), && \text{for all } i \in I \text{ with } \#(i) = 0,
\end{aligned}$$

where we may assume, without loss of generality, that $X^0 = Y^0 = \{*\}$. It is easy to see that $F_\#$ is now actually a functor.

Now we check that the assignment $\# \mapsto F_\#$ extends to a functor from $\mathbf{Sets}/\mathbb{N}$ to $\mathrm{Endo}(\mathbf{Sets})$. Suppose we are given a pair of arities $\#_1\colon I_1 \to \mathbb{N}$ and $\#_2\colon I_2 \to \mathbb{N}$ and a morphism $(I_1, \#_1) \xrightarrow{f} (I_2, \#_2)$

in **Sets**/$\mathbb{N}$. This f is a set function $f\colon I_1 \to I_2$ making the diagram

$$\begin{tikzcd} I_1 \arrow[rr, "f"] \arrow[dr, "\#_1"'] & & I_2 \arrow[dl, "\#_2"] \\ & \mathbb{N} & \end{tikzcd}$$

in **Sets** commute. We define a natural transformation $F_{\#_1} \overset{\alpha^f}{\Longrightarrow} F_{\#_2}$ as follows: for a set X, the function $\alpha^f_X\colon F_{\#_1}X \to F_{\#_2}X$ is defined to be the mapping

$$F_{\#_1}X = \coprod_{j\in I_1} X^{\#_1(j)} \ni (i,x) \xmapsto{\alpha^f_X} (f(i),x) \in \coprod_{j\in I_2} X^{\#_2(j)} = F_{\#_2}X$$

This α^f is indeed a natural transformation: for a morphism $g\colon X \to Y$ in **Sets**, the diagram

$$\begin{tikzcd} F_{\#_1}X \arrow[r, "F_{\#_1}g"] \arrow[d, "\alpha^f_X"'] & F_{\#_1}Y \arrow[d, "\alpha^f_Y"] \\ F_{\#_2}X \arrow[r, "F_{\#_2}g"] & F_{\#_2}Y \end{tikzcd}$$

in **Sets** commutes because:

1. we have

$$\begin{aligned}(\alpha^f_Y \circ F_{\#_1}g)(i,x) &= \alpha^f_Y\big(i,(g(x_1),\dots,g(x_{\#_1(i)}))\big)\\ &= \big(f(i),(g(x_1),\dots,g(x_{\#_1(i)}))\big)\\ &= (F_{\#_2}g)\big(f(i),(x_1,\dots,x_{\#_1(i)})\big)\\ &= ((F_{\#_2})\circ\alpha^f_Y)\big(i,(x_1,\dots,x_n)\big)\end{aligned}$$

   for all $i \in I_1$ with $\#_1(i) \neq 0$ and for all $x = (x_1,\dots,x_{\#_1(i)}) \in X^{\#_1(i)}$;

2. and we have

$$\begin{aligned}(\alpha^f_Y \circ F_{\#_1}g)(i,*) &= \alpha^f_Y(i,*)\\ &= (f(i),*)\\ &= (F_{\#_2}g)(f(i),*)\\ &= (F_{\#_2}g)\circ\alpha^f_Y)(i,*)\end{aligned}$$

   for all $i \in I_1$ with $\#_1(i) = 0$.

It is then easy to see that the assignment $\mathbf{Sets}/\mathbb{N} \to \mathrm{Endo}(\mathbf{Sets})$, defined on objects by $\mathbf{Sets}/\mathbb{N} \ni \# \mapsto F_\# \in \mathrm{Endo}(\mathbf{Sets})$ and on morphisms by $\mathsf{Arr}(\mathbf{Sets}/\mathbb{N}) \ni f \mapsto \alpha^f \in \mathsf{Arr}(\mathrm{Endo}(\mathbf{Sets}))$, is a functor. This is because compositions of natural transformations are simply their pointwise compositions. $\square$

Exercise 2.5.4

This exercise describes 'strength' for endofunctors on **Sets**. *In general, this is a useful notion in the theory of datatypes (Cockett and Spencer, 1992), (Cockett and Spencer, 1995) and of computations (Moggi, 1991); see Section 5.2 for a systemic description.*

Let $F\colon \mathbf{Sets} \to \mathbf{Sets}$ be an arbitrary functor. Consider for sets X, Y the strength map

$$\begin{array}{ccc} F(X) \times Y & \xrightarrow{\quad \mathrm{st}_{X,Y} \quad} & F(X \times Y) \\ \\ (u, y) & \longmapsto & F(\lambda x \in X.(x, y))(u) \end{array}$$

1. *Prove that this yields a natural transformation $F(-)\times(-) \overset{\mathrm{st}}{\Rightarrow} F((-)\times(-))$, where both the domain and codomain are functors $\mathbf{Sets} \times \mathbf{Sets} \to \mathbf{Sets}$.*
2. *Describe this strength map for the list functor $(-)^\star$ and for the powerset functor $\mathcal{P}$.*

Solution.

1. Suppose we are given two functions $f\colon X \to A$ and $g\colon Y \to B$. We have to show that the square

$$\begin{tikzcd} FX \times Y \arrow[r, "\mathrm{st}_{X,Y}"] \arrow[d, "Ff\times g"'] & F(X \times Y) \arrow[d, "F(f\times g)"] \\ FA \times B \arrow[r, "\mathrm{st}_{A,B}"'] & F(A \times B) \end{tikzcd}$$

in $\mathbf{Sets}$ commutes. Indeed, for any $(u, y) \in FX \times Y$, we have

$$\begin{aligned} (F(f\times g)\circ \mathrm{st}_{X,Y})(u,y) &= F(f\times g)\big(F(\lambda x \in X.(x,y))(u)\big) \\ &= \big(F(f\times g)\circ F(\lambda x \in X.(x,y))\big)(u) \\ &= F\big((f\times g)\circ(\lambda x \in X.(x,y))\big)(u) \\ &= F\big(\lambda x \in X.(f(x), g(y))\big)(u) \\ &= F\Big(\big(\lambda a \in A.(a, g(y))\big)\circ f\Big)(u) \\ &= \Big(F\big(\lambda a \in A.(a, g(y))\big)\circ Ff\Big)(u) \\ &= F\big(\lambda a \in A.(a, g(y))\big)\big((Ff)(u)\big) \\ &= \mathrm{st}_{A,B}\big((Ff)(u),\, g(y)\big) \\ &= (\mathrm{st}_{A,B}\circ Ff\times g)(u,y). \end{aligned}$$

2. Recall that the list functor $(-)^\star\colon \mathbf{Sets} \to \mathbf{Sets}$ sends a set X to the set $\bigcup_{n\in\mathbb{N}} X^n$ of all finite sequences of elements of X, and sends a function $f\colon X \to Y$ to the function $f^\star\colon X^\star \to Y^\star$ which maps a sequence $\langle x_0, \ldots, x_n\rangle \in X^\star$ to the sequence $\langle f(x_0), \ldots, f(x_n)\rangle \in Y^\star$ (and $f^\star$ maps the empty sequence to the empty sequence). The strength map $\mathrm{st}^{\mathrm{list}}\colon (-)^\star \times (-) \Rightarrow ((-)\times(-))^\star$ for the list functor thus satisfies

$$\mathrm{st}^{\mathrm{list}}_{X,Y}(\langle\rangle, y) = \langle\rangle \quad \text{and} \quad \mathrm{st}^{\mathrm{list}}_{X,Y}(\langle x_0,\ldots,x_n\rangle, y) = \langle (x_0,y),\ldots,(x_n,y)\rangle$$

for all sets X and Y, all finite sequences $\langle x_0, \ldots, x_n\rangle \in X^\star$, and all elements $y \in Y$.

On the other hand, the strength map $\mathrm{st}^{\mathrm{powerset}}\colon \mathcal{P}(-)\times(-) \to \mathcal{P}((-)\times(-))$ for the power set functor satisfies

$$\mathrm{st}^{\mathrm{powerset}}_{X,Y}(U, y) = \{\, (x,y) : x \in U \,\}$$

for all sets X and Y, all subsets $U \subseteq X$, and all elements $y \in Y$. □

Exercise 2.5.5
The following strengthening of induction is sometimes called 'induction with parameters'. It is different from recursion, which also involves an additional parameter; see Proposition 2.4.7.

Assume a functor F with a strength natural transformation as in the previous exercise and with initial algebra $\alpha\colon F(A) \xrightarrow{\cong} A$. Let P be a set (or object) for parameters. Prove that for each map $h\colon F(X) \times P \to X$ there is a unique $f\colon A \times P \to X$ making the following diagram commute:

$$\begin{tikzcd}
F(A) \times P \arrow[r, "\langle F(f)\circ \mathrm{st}, \pi_2\rangle"] \arrow[d, "\alpha\times\mathrm{id}"', "\cong"] & F(X) \times P \arrow[d, "h"] \\
A \times P \arrow[r, "f"'] & X
\end{tikzcd}$$

Hint: First turn h into a suitable algebra $h'\colon F(X^P) \to X^P$ via strength.

Use this mechanism to define the append map $\mathsf{app}\colon A^\star \times A^\star \to A^\star$ *from Example 2.4.6.*

Solution. Suppose we are given a functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$, an initial F-algebra $\alpha\colon FA \to A$, a strength natural transformation $\mathrm{st}\colon F(-) \times (-) \Rightarrow F((-) \times (-))$, a set P, and a function $h\colon F(X) \times P \to X$.

For sets Y and Z, let $Y \xleftarrow{\pi_1^{Y\times Z}} Y \times Z \xrightarrow{\pi_2^{Y\times Z}} Z$ denote the relevant projection functions and let $Z^Y \times Y \xrightarrow{\mathrm{ev}_Z^Y} Y$ denote the relevant evaluation function. For sets W, Y, Z and a function $g\colon W \times Y \to Z$, let $\overline{g}\colon W \to Z^Y$ be the unique function such that $\mathrm{ev}_Z^Y \circ (\overline{g} \times \mathrm{id}_P) = g$.

We have the commuting diagram

$$\begin{tikzcd}
FX & X & P \\
F(X^P \times P) \arrow[u, "F(\mathrm{ev}_X^P)"] & FX \times P \arrow[ul, "\pi_1^{FX\times P}"'] \arrow[u, "h"] \arrow[ur, "\pi_2^{FX\times P}"] & \\
 & F(X^P) \times P \arrow[ul, "\mathrm{st}_{X^P,P}"] \arrow[u, dashed, "\exists! k"] \arrow[uur, bend right, "\pi_2^{F(X^P)\times P}"'] &
\end{tikzcd}$$

in **Sets**, which gives rise to a unique function $f\colon A \times X \to P$ making the square

$$\begin{tikzcd}
FA \arrow[r, "F\overline{f}"] \arrow[d, "\alpha"'] & F(X^P) \arrow[d, "\overline{hk}"] \\
A \arrow[r, "\overline{f}"'] & X^P
\end{tikzcd}$$

in **Sets** commute, by the initiality of (A, α) in $\mathbf{Alg}(F)$. We wish to show that the diagram

$$\begin{tikzcd}
 & & P & \\
FA \times P \arrow[urr, "\pi_2^{FA\times P}"] \arrow[rrr, "\left\langle (Ff)\mathrm{st}_{A,P},\, \pi_2^{FA\times P}\right\rangle"] \arrow[dr, "\mathrm{st}_{A,P}"'] \arrow[dd, "\alpha\times\mathrm{id}"', "\cong"] & & & FX \times P \arrow[ul, "\pi_2^{FX\times P}"'] \arrow[dl, "\pi_1^{FX\times P}"] \arrow[dd, "h"] \\
 & F(A \times P) \arrow[r, "Ff"'] & FX & \\
A \times P \arrow[rrr, "f"'] & & & X
\end{tikzcd}$$

in **Sets** commutes.

Our solution to Exercise 2.1.8 shows that the functor $(-)\times P\colon \mathbf{Sets}\to\mathbf{Sets}$ is left adjoint to the functor $(-)^P\colon \mathbf{Sets}\to\mathbf{Sets}$, so that we have the commuting diagram

$$\begin{tikzcd}
\hom_{\mathbf{Sets}}(FA\times P, X) & & \hom_{\mathbf{Sets}}(FA, X^P) \arrow[ll, "(-)\times P\,\dashv\,(-)^P", "\cong"'] \\
\hom_{\mathbf{Sets}}(A\times P, X) \arrow[u, "(-)\circ(\alpha\times\mathrm{id}_P)"] & & \hom_{\mathbf{Sets}}(A, X^P) \arrow[ll, "\cong", "(-)\times P\,\dashv\,(-)^P"'] \arrow[u, "(-)\circ\alpha"']
\end{tikzcd}$$

in **Sets**. The adjunction bijection takes, for instance, a function $\overline{g}\in\hom_{\mathbf{Sets}}(A, X^P)$ and sends it to the function $\mathrm{ev}_X^P\circ(\overline{g}\times\mathrm{id}_P)\in\hom_{\mathbf{Sets}}(A\times P, X)$. Going along the bottom and then the left morphisms of the diagram above takes the function $\overline{f}\in\hom_{\mathbf{Sets}}(A, X^P)$ to the function $f\circ(\alpha\times\mathrm{id}_P)\in\hom_{\mathbf{Sets}}(FA\times P, X)$. On the other hand, the right morphism of the diagram above will take $\overline{f}$ to the function $\overline{f}\alpha$, which is equal to $\overline{hk}\circ F\overline{f}$. The top morphism of the diagram above then sends $\overline{hk}\circ F\overline{f}$ to $\mathrm{ev}_X^P\circ((\overline{hk}\circ F\overline{f})\times\mathrm{id}_P)$. We are thus reduced to showing that $\mathrm{ev}_X^P\circ((\overline{hk}\circ F\overline{f})\times\mathrm{id}_P) = h\circ\left\langle (Ff)\mathrm{st}_{A,P},\, \pi_2^{FA\times P}\right\rangle$. This follows from the commutativity of the diagrams

$$\begin{tikzcd}
 & X^P\times P \arrow[rr, "\mathrm{ev}_X^P"] & & X \\
FA\times P \arrow[r, "F\overline{f}\times\mathrm{id}_P"] \arrow[d, "\mathrm{st}_{A,P}"'] & F(X^P)\times P \arrow[u, "\overline{hk}\times\mathrm{id}_P"] \arrow[r, "k"'] \arrow[d, "\mathrm{st}_{X^P,P}"] & FX\times P \arrow[ur, "h"'] \arrow[ddl, "\pi_1^{FX\times P}"] & \\
F(A\times P) \arrow[r, "F(\overline{f}\times\mathrm{id}_P)"] \arrow[dr, "Ff"'] & F(X^P\times P) \arrow[d, "F(\mathrm{ev}_X^P)"] & & \\
 & FX & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
FA\times P \arrow[r, "F\overline{f}\times\mathrm{id}_P"] \arrow[dr, "\pi_2^{FA\times P}"'] & F(X^P)\times P \arrow[r, "k"] \arrow[d, "\pi_2^{F(X^P)\times P}"] & FX\times P \arrow[dl, "\pi_2^{FX\times P}"] \\
 & P &
\end{tikzcd}$$

in **Sets**. The uniqueness of f follows from the adjunction bijection and its naturality: for any $f'\in\hom_{\mathbf{Sets}}(A\times P, X)$ satisfying the equality $f'\circ(\alpha\times\mathrm{id}_P) = h\circ\left\langle (Ff')\mathrm{st}_{A,P},\, \pi_2^{FA\times P}\right\rangle$, the function $\overline{f'}\in\hom_{\mathbf{Sets}}(A, X^P)$ must also be an F-algebra homomorphism from (A,α) to $(X^P,\overline{hk})$.

Now consider the case where $F = 1+A\times(-)\colon \mathbf{Sets}\to\mathbf{Sets}$, whose initial algebra is $\alpha\colon 1+(A\times A^\star)\to A^\star$ defined by $\alpha(*) := \langle\rangle$, where $*$ is the unique element in 1, and $\alpha(a,\sigma) := a\cdot\sigma$ for all $(a,\sigma)\in A\times A^\star$. For sets X and Y, the associated strength map $\mathrm{st}_{X,Y}\colon (1+(A\times X))\times Y\to 1+A\times(X\times Y)$ from Exercise 2.5.4 is defined by

$$\mathrm{st}_{X,Y}(*,y) := *,$$
$$\mathrm{st}_{X,Y}((a,x),y) := (a,(x,y)),$$

for all $a \in A$, $x \in X$, and $y \in Y$, where we have assumed without loss of generality that all coproducts are simply disjoint unions.

Define $h\colon (1 + (A \times A^\star)) \times A^\star \to A^\star$ by $h(*, \tau) \coloneqq \tau$ for all $\tau \in A^\star$, and $h((a, \sigma), \tau) \coloneqq a \cdot \sigma$ for all $((a, \sigma), \tau) \in (A \times A^\star) \times A^\star$. The result above tells us that we have the commuting diagram

$$\begin{tikzcd}
(1 + (A \times A^\star)) \times A^\star \arrow[r, dashed, "\left\langle (\mathrm{id}_1 + (\mathrm{id}_A \times \mathsf{app})) \circ \mathrm{st}_{A,A^\star},\ \pi_2^{(1+(A\times A))\times A^\star} \right\rangle"] \arrow[d, "\alpha"'] & (1 + (A \times A^\star)) \times A^\star \arrow[d, "h"] \\
A^\star \times A^\star \arrow[r, dashed, "\exists!\mathsf{app}"'] & A^\star
\end{tikzcd}$$

in **Sets**. We claim that the unique induced function $\mathsf{app}\colon A^\star \times A^\star \to A^\star$ is our desired append function. For any $\tau \in A^\star$, the diagram above gives the chase of elements

$$\begin{tikzcd}
(*, \tau) \arrow[r, mapsto, "\left\langle (\mathrm{id}_1 + (\mathrm{id}_A \times \mathsf{app})) \circ \mathrm{st}_{A,A^\star},\ \pi_2^{(1+(A\times A))\times A^\star} \right\rangle"] \arrow[d, mapsto, "\alpha"'] & (*, \tau) \arrow[d, mapsto, "h"] \\
(\langle\rangle, \tau) & \tau
\end{tikzcd}$$

and so $\mathsf{app}(\langle\rangle, \tau) = \tau$. Furthermore, for any $a \in A$ and $\sigma, \tau \in A^\star$, we get the chase of elements

$$\begin{tikzcd}
((a, \sigma), \tau) \arrow[r, mapsto, "\left\langle (\mathrm{id}_1 + (\mathrm{id}_A \times \mathsf{app})) \circ \mathrm{st}_{A,A^\star},\ \pi_2^{(1+(A\times A))\times A^\star} \right\rangle"] \arrow[d, mapsto, "\alpha"'] & ((a, \mathsf{app}(\sigma, \tau)), \tau) \arrow[d, mapsto, "h"] \\
(a \cdot \sigma, \tau) & a \cdot \mathsf{app}(\sigma, \tau)
\end{tikzcd}$$

and so $\mathsf{app}(a \cdot \sigma, \tau) = a \cdot \mathsf{app}(\sigma, \tau)$. □

Exercise 2.5.6

Show that the forgetful functor $U\colon \mathbf{Sp} \to \mathbf{Sets}$ from topoligcal spaces to sets has both a left adjoint (via the discrete topology on a set, in which every subset is open) and a right adjoint (via the indiscrete topology, with only the empty set and the whole set itself as open sets).

Solution. Fix a pair of sets X and Y. If X is equipped with the discrete topology, then for any topology on Y, every function $f\colon X \to Y$ is continuous. On the other hand, if Y is equipped with the indiscrete topology, then for any topology on X, every function $f\colon X \to Y$ is continuous.

So, for any set X and any topological space T, we have bijections

$$\hom_{\mathbf{Sp}}(DX, T) \xrightarrow[\cong]{\varphi_{X,T}} \hom_{\mathbf{Sets}}(X, UT) \qquad \hom_{\mathbf{Sets}}(UT, X) \xrightarrow[\cong]{\psi_{X,T}} \hom_{\mathbf{Sets}}(T, IX)$$

where $D\colon \mathbf{Sets} \to \mathbf{Sp}$ is the discrete topology functor and $I\colon \mathbf{Sets} \to \mathbf{Sp}$ is the indiscrete topology functor, both of which are send a function to itself considered as a continuous map. The bijections above would send each function or continuous map to itself. The naturality of φ and ψ in both X and T is then immediate because D, I, $\varphi_{X,T}$, and $\psi_{X,T}$ all send a morphism to itself. □

Exercise 2.5.7
Assume two functor $F\colon \mathbb{C} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{C}$ in opposite directions, with a natural transformation $\eta\colon \mathrm{id}_C \Rightarrow GF$. Define functions $\psi\colon \mathbb{D}(F(X), Y) \to \mathbb{C}(X, G(Y))$ by $\psi(f) = G(f) \circ \eta_X$.

1. *Check that such ψs are natural.*

2. *Prove that these ψs are isomorphisms if and only if there exists a natural transformation $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{D}}$ such that the following **triangular identities** hold:*

$$G(\varepsilon) \circ \eta_G = \mathrm{id}, \qquad \varepsilon_F \circ F(\eta) = \mathrm{id}.$$

Solution. Write $\psi_{X,Y}$ for the function from $\hom_{\mathbb{D}}(FX, Y)$ to $\hom_{\mathbb{C}}(X, GY)$ defined by $\psi_{X,Y}(h) \coloneqq Gh \circ \eta_X$ for all $h \in \hom_{\mathbb{D}}(FX, Y)$.

1. Suppose we are given a morphism $X \xleftarrow{f} X'$ in $\mathbb{C}$ and a morphism $Y \xrightarrow{g} Y'$ in $\mathbb{D}$. We wish to show that the diagram

$$\begin{tikzcd}
\hom_{\mathbb{D}}(FX, Y) \arrow[r, "\psi_{X,Y}"] \arrow[d, "g \circ (-) \circ Ff"'] & \hom_{\mathbb{C}}(X, GY) \arrow[d, "Gg \circ (-) \circ f"] \\
\hom_{\mathbb{D}}(F(X'), Y') \arrow[r, "\psi_{X',Y'}"'] & \hom_{\mathbb{C}}(X', G(Y'))
\end{tikzcd}$$

commutes. Indeed, for any $h \in \hom_{\mathbb{D}}(FX, Y)$, we have

$$\begin{aligned}
Gg \circ \psi_{X,Y}(h) \circ f &= Gg \circ Gh \circ \eta_X \circ f \\
&= G(gh) \circ \eta_X \circ f \\
&= G(gh) \circ GFf \circ \eta_{X'}, \quad \text{by the naturality of } \eta, \\
&= G(g \circ h \circ Ff) \circ \eta_{X'} \\
&= \psi_{X',Y'}(g \circ h \circ Ff).
\end{aligned}$$

2. Suppose that $\psi_{X,Y}\colon \hom_{\mathbb{D}}(FX, Y) \to \hom_{\mathbb{C}}(X, GY)$ is an isomorphism for every $X \in \mathbb{C}$ and $Y \in \mathbb{D}$, so that ψ is a natural isomorphism from $\hom_{\mathbb{D}}(F(-), -)$ to $\hom_{\mathbb{C}}(-, G(-))$, establishing an adjunction between F and G. So we can define a natural transformation $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{D}}$ by $\varepsilon_Y \coloneqq \psi^{-1}_{GY,Y}(\mathrm{id}_{GY})$ for all $Y \in \mathbb{D}$. We should check that this is natural in Y. Indeed, for any morphism $Y \xrightarrow{g} Y'$ in $\mathbb{D}$, the commuting diagram on the left

$$\begin{tikzcd}
\hom_{\mathbb{D}}(FGY, Y) \arrow[r, "\psi_{GY,Y}", "\cong"'] \arrow[d, "g \circ (-)"'] & \hom_{\mathbb{C}}(GY, GY) \arrow[d, "Gg \circ (-)"] \\
\hom_{\mathbb{D}}(FGY, Y') \arrow[r, "\psi_{GY,Y'}", "\cong"'] & \hom_{\mathbb{C}}(GY, GY') \\
\hom_{\mathbb{D}}(FGY', Y') \arrow[r, "\cong", "\psi_{GY',Y'}"'] \arrow[u, "(-) \circ FGg"] & \hom_{\mathbb{C}}(GY', GY') \arrow[u, "(-) \circ Gg"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\varepsilon_Y \arrow[r, mapsto, "\psi_{GY,Y}", "\cong"'] \arrow[d, mapsto, "g \circ (-)"'] & \mathrm{id}_{GY} \arrow[d, mapsto, "Gg \circ (-)"] \\
\psi^{-1}_{GY,Y'}(Gg) \arrow[r, mapsto, "\psi_{GY,Y'}", "\cong"'] & Gg \\
\varepsilon_{Y'} \arrow[r, mapsto, "\cong", "\psi_{GY',Y'}"'] \arrow[u, mapsto, "(-) \circ FGg"] & \mathrm{id}_{GY'} \arrow[u, mapsto, "(-) \circ Gg"']
\end{tikzcd}$$

yields the chase of elements on the right, from which we conclude that $g\varepsilon_Y = \varepsilon_{Y'}FGg$. Thus ε really is a natural transformation from FG to $\mathrm{id}_{\mathbb{D}}$.

It remains to check that this ε (together with η) satisfy the triangular identities. The first triangular identity follows from the following quick calculation: for all $Y \in \mathbb{D}$, we have

$$G\varepsilon_Y \circ \eta_{GY} = \psi_{GY,Y}(\varepsilon_Y) = \mathrm{id}_{GY}$$

where the second equality follows from the definition of ε_Y. For the second triangular identity, we see that for $X \in \mathbb{C}$ we have the commuting diagram on the left

$$\begin{tikzcd}
\hom_{\mathbb{D}}(FGFX, FX) \arrow[r, "\psi_{GFX,X}", "\cong"'] \arrow[d, "(-)\circ F\eta_X"'] & \hom_{\mathbb{C}}(GFX, GFX) \arrow[d, "(-)\circ \eta_X"] \\
\hom_{\mathbb{D}}(FX, FX) \arrow[r, "\cong", "\psi_{X,FX}"'] & \hom_{\mathbb{C}}(X, GFX)
\end{tikzcd}
\qquad
\begin{tikzcd}
\varepsilon_{FX} \arrow[r, mapsto, "\psi_{GFX,X}", "\cong"'] \arrow[d, mapsto, "(-)\circ F\eta_X"'] & \mathrm{id}_{GFX} \arrow[d, mapsto, "(-)\circ \eta_X"] \\
\psi^{-1}_{X,FX}(\eta_X) \arrow[r, mapsto, "\cong", "\psi_{X,FX}"'] & \eta_X
\end{tikzcd}$$

from which we obtain the chase of elements on the right. From this, we conclude that

$$\varepsilon_{FX} F\eta_X = \psi^{-1}_{X,FX}(\eta_X) = \mathrm{id}_{FX}$$

since $\psi_{X,FX}$ is an isomorphism and $\psi_{X,FX}(\mathrm{id}_{FX}) = G\mathrm{id}_{FX} \circ \eta_X = \eta_X$.

Conversely, suppose that there is a natural transformation $\varepsilon \colon FG \Rightarrow \mathrm{id}_{\mathbb{D}}$ such that the triangular identities hold. For any $X \in \mathbb{C}$ and $Y \in \mathbb{D}$, we claim that the mappings defined by

$$\begin{tikzcd}
h \arrow[r, mapsto] & Gh \circ \eta_X \\
\hom_{\mathbb{D}}(FX, Y) \arrow[u, phantom, "\Cap"] \arrow[r, shift left, "\psi_{X,Y}"] & \hom_{\mathbb{C}}(X, GY) \arrow[u, phantom, "\Cap"] \arrow[l, shift left, "\varphi_{X,Y}"] \\
\varepsilon_Y \circ Fk \arrow[u, phantom, "\Cup"] & k \arrow[l, mapsto] \arrow[u, phantom, "\Cup"]
\end{tikzcd}$$

are inverses to each other. Indeed, for any $h \in \hom_{\mathbb{D}}(FX, Y)$, we have

$$\begin{aligned}
\varphi_{X,Y}(\psi_{X,Y}(h)) &= \varphi_{X,Y}(Gh \circ \eta_X) \\
&= \varepsilon_Y \circ F(Gh \circ \eta_X) \\
&= \varepsilon_Y \circ FGh \circ F\eta_X \\
&= h \circ \varepsilon_{FX} \circ F\eta_X \\
&= h,
\end{aligned}$$

where the second last line follows from the naturality of ε, and the last line follows from the second triangular identity. Similarly, for any $k \in \hom_{\mathbb{C}}(X, GY)$, we have

$$\begin{aligned}
\psi_{X,Y}(\varphi_{X,Y}(k)) &= \psi_{X,Y}(\varepsilon_Y \circ Fk) \\
&= G(\varepsilon_Y \circ Fk) \circ \eta_X \\
&= G\varepsilon_Y \circ GFk \circ \eta_X \\
&= G\varepsilon_Y \circ \eta_{GY} \circ k \\
&= k.
\end{aligned}$$

In the case when such an ε exists as in part (2), we say that η and ε are, respectively, the **unit** and **counit** of the adjunction $F \dashv G$. ☐

Exercise 2.5.8

A morphism $m\colon X' \to X$ in a category $\mathbb{D}$ is called a ***monomorphism*** *(or a* ***mono****, for short), written as $m\colon X' \rightarrowtail X$, if for each parallel pair of arrows $f, g\colon Y \to X'$, $m \circ f = m \circ g$ implies $f = g$.*

1. *Prove that the monomorphisms in* **Sets** *are precisely the injective functions.*
2. *Let $G\colon \mathbb{D} \to \mathbb{C}$ be a right adjoint. Show that if m is a monomorphism in $\mathbb{D}$, then so is $G(m)$ in $\mathbb{C}$.*

Dually, an ***epimorphism*** *(or* ***epi****, for short) in $\mathbb{C}$ is an arrow written as $e\colon X' \twoheadrightarrow X$ such that for all maps $f, g\colon X \to Y$, if $f \circ e = g \circ e$ then $f = g$.*

3. *Show that the epimorphisms in* **Sets** *are the surjective functions. Hint: For an epi $X \twoheadrightarrow Y$, choose two appropriate maps $Y \to 1 + Y$.*
4. *Prove that left adjoints preserve epimorphisms.*

Solution.

1. Suppose $m\colon X' \rightarrowtail X$ is a monomorphism in **Sets**. So, for any pair of functions $f, g\colon \{*\} \to X'$ with $mf = mg$, we have $f = g$. Functions $\{*\} \to X'$ in **Sets** correspond to elements of X'. So $m\colon X' \to X$ is injective.

   Conversely, suppose $m\colon X' \to X$ is injective. Fix a pair of functions $f, g\colon Y \to X'$ satisfying $mf = mg$. Then, for every $y \in Y$, we have $m(f(y)) = m(g(y))$, and so $f(y) = g(y)$ since m is injective. So $f = g$.

2. Suppose we have an adjunction $\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$ and a monomorphism $X \overset{m}{\rightarrowtail} Y$ in $\mathbb{D}$. Fix a parallel pair of morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} GX$ in $\mathbb{C}$ satisfying $(Gm)f = (Gm)g$. Let $\psi\colon \hom_{\mathbb{D}}(F(-),(-)) \xrightarrow{\cong} \hom_{\mathbb{C}}((-), G(-))$ denote the adjunction bijection. Let $\bar{f} := \psi^{-1}_{A,X}(f)$ and $\bar{g} := \psi^{-1}_{A,X}(g)$ denote the morphisms which correspond to f and g respectively under the adjunction bijection. The naturality of ψ gives commuting diagram below on the left (which will live in **Sets** if both $\mathbb{C}$ and $\mathbb{D}$ are locally small)

$$\begin{tikzcd} \hom_{\mathbb{C}}(A, GX) \arrow[d, "(Gm)\circ(-)"'] & \hom_{\mathbb{D}}(FA, X) \arrow[l, "\psi_{A,X}"', "\cong"] \arrow[d, "m\circ(-)"] \\ \hom_{\mathbb{C}}(A, GY) & \hom_{\mathbb{D}}(FA, Y) \arrow[l, "\cong"', "\psi_{A,Y}"] \end{tikzcd} \qquad \begin{tikzcd} f, g \arrow[d, mapsto] & \bar{f}, \bar{g} \arrow[l, mapsto] \arrow[d, mapsto] \\ (Gm)f = (Gm)g & m\bar{f} = m\bar{g} \arrow[l, mapsto] \end{tikzcd}$$

   from which we obtain the chase of elements above on the right. As m is a monomorphism, we obtain $\bar{f} = \bar{g}$. Therefore $f = g$ since $\psi_{A,X}\colon \hom_{\mathbb{D}}(FA, X) \to \hom_{\mathbb{C}}(A, GX)$ is bijective. Therefore $GX \xrightarrow{Gm} GY$ is a monomorphism in $\mathbb{C}$.

3. Suppose $e\colon X' \twoheadrightarrow X$ is an epimorphism in **Sets**. If $e\colon X' \to X$ is not surjective, then the two functions $f, g\colon X \to \{*\} + X$ defined by

$$f(x) := \begin{cases} x, & \text{if } x \in \mathrm{Img}(e), \\ *, & \text{if } x \notin \mathrm{Img}(e), \end{cases} \quad \text{and} \quad g(x) := x$$

for all $x \in X$ satisfy $fe = ge$. As e is epic, we have $f = g$. Therefore we must have $\mathrm{Img}(e) = X$, i.e. $e\colon X' \to X$ is surjective.

Conversely, suppose that $e\colon X' \to X$ is surjective. Fix a pair of functions $f, g\colon X \to Y$ satisfying $fe = ge$. Then, for any $x \in X$, there exists $x' \in X'$ with $e(x') = x$, and so $f(x) = f(e(x')) = g(e(x')) = g(x)$. So $f = g$.

4. The proof is dual to the proof of part (2) of this exercise. Fix an adjunction $\mathbb{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathbb{D}$, an epimorphism $X \overset{e}{\twoheadrightarrow} Y$ in $\mathbb{C}$, and a parallel pair of morphisms $FY \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathbb{D}$ satisfying $f(Fe) = g(Fe)$. Define $\bar{f} \coloneqq \psi_{Y,B}(f)$ and $\bar{g} \coloneqq \psi_{Y,B}(g)$, where $\psi\colon \hom_{\mathbb{D}}(F(-),(-)) \xrightarrow{\cong} \hom_{\mathbb{C}}((-),G(-))$ is the adjunction bijection. Staring at the commuting diagrams

$$\begin{tikzcd}
\hom_{\mathbb{C}}(Y, GB) \arrow[d, "(-)\circ e"'] & \hom_{\mathbb{D}}(FY, B) \arrow[l, "\psi_{Y,B}"', "\cong"] \arrow[d, "(-)\circ(Fe)"] \\
\hom_{\mathbb{C}}(X, GB) & \hom_{\mathbb{D}}(FX, B) \arrow[l, "\cong"', "\psi_{X,B}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{f}, \bar{g} \arrow[d, mapsto] & f, g \arrow[l, mapsto] \arrow[d, mapsto] \\
\bar{f} \circ e = \bar{g} \circ e & f(Fe) = g(Fe) \arrow[l, mapsto]
\end{tikzcd}$$

yields $f = g$. $\square$

Exercise 2.5.9
Notice that the existence of final coalgebras for finite polynomial functors (Theorem 2.3.9) that is used in the proof of Proposition 2.5.3 is actually a special case of this proposition. Hint: Consider the right adjoint to the final set 1.

Solution. Recall that Proposition 2.5.3 asserts that for every finite Kripke polynomial functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$, the forgetful functor $U\colon \mathbf{CoAlg}(F) \to \mathbf{Sets}$ has a right adjoint, say, $G\colon \mathbf{Sets} \to \mathbf{CoAlg}(F)$. We claim that $G\colon \mathbf{Sets} \to \mathbf{CoAlg}(F)$ preserves the terminal object $1 \in \mathbf{Sets}$. Indeed, for any F-coalgebra $A \xrightarrow{\alpha} FA$, we have the bijections

$$\hom_{\mathbf{CoAlg}(F)}\big((A,\alpha),\ G1\big) \cong \hom_{\mathbf{Sets}}\big(U(A,\alpha),\ 1\big) = \hom_{\mathbf{Sets}}(A, 1)$$

and we know that $\hom_{\mathbf{Sets}}(A, 1)$ has exactly one element. $\square$

Exercise 2.5.10
Assume an endofunctor $F\colon \mathbb{C} \to \mathbb{C}$. Prove that there are natural transformations

$$\begin{tikzcd}
\mathbf{CoAlg}(F) \arrow[r, bend left=50, "U"{name=A}] \arrow[r, bend right=50, "FU"'{name=B}] & \mathbb{C} \arrow[from=A, to=B, Rightarrow]
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathbf{Alg}(F) \arrow[r, bend left=50, "FU"{name=A}] \arrow[r, bend right=50, "U"'{name=B}] & \mathbb{C} \arrow[from=A, to=B, Rightarrow]
\end{tikzcd}$$

where U is the forgetful functor. (In 2-categorical terminology these maps form inserters; see e.g. Hermida and Jacobs (1998, appendix).)

Solution. Let $U\colon \mathbf{CoAlg}(F) \to \mathbb{C}$ denote the relevant forgetful functor. Define a natural transformation $U \overset{\eta}{\Rightarrow} FU$ as follows: for $(X, c) \in \mathbf{CoAlg}(F)$, define $\eta_{(X,c)} \coloneqq c$. We have to check that this is natural in (X, α). Suppose we are given a homomorphism of F-coalgebras $(X, c) \xrightarrow{f} (Y, d)$. Then the diagram

$$\begin{tikzcd}
FU(X,c) = FX \arrow[r, "FUf = Ff"] & FU(Y,d) = FY \\
U(X,c) = X \arrow[u, "\eta_{(X,c)} = c"] \arrow[r, "Uf = f"'] & U(Y,d) = Y \arrow[u, "\eta_{(Y,d)} = d"']
\end{tikzcd}$$

in $\mathbb{C}$ commutes since f is a homomorphism of F-coalgebras.

Similarly, if $U'\colon \mathbf{Alg}(F) \to \mathbb{C}$ denotes the relevant forgetful functor, then we can define a natural transformation $FU \overset{\xi}{\Rightarrow} U$ by $\xi_{(X,a)} \coloneqq a$ for all $(X, a) \in \mathbf{Alg}(F)$. The naturality of ξ will then follow from the fact that morphisms in $\mathbf{Alg}(F)$ are F-algebra homomorphisms. $\square$

Exercise 2.5.11 (Hughes)
Let $\mathbb{C}$ be an arbitrary categroy with products $\times$ and let $F, H\colon \mathbb{C} \to \mathbb{C}$ be two endofunctors on $\mathbb{C}$. Assume that cofree F-coalgebras exist, i.e. that the forgetful functor $U\colon \mathbf{CoAlg}(F) \to \mathbb{C}$ has a right adjoint G — as in Proposition 2.5.3. Prove then that there is an isomorphism of categories of coalgebras

$$\mathbf{CoAlg}(F \times H) \xrightarrow{\quad\cong\quad} \mathbf{CoAlg}(GHU)$$

where $\mathbf{CoAlg}(GHU)$ is a category of coalgebras on coalgebras, for the functor composition $GHU\colon \mathbf{CoAlg}(F) \to \mathbb{C} \to \mathbb{C} \to \mathbf{CoAlg}(F)$.

Solution. For objects $X, Y \in \mathbb{C}$, let $X \xleftarrow{\pi_1^{X\times Y}} X \times Y \xrightarrow{\pi_2^{X\times Y}} Y$ denote the projection morphisms associated with the product $X \times Y$ in $\mathbb{C}$. Furthermore, for an F-coalgebra (X, c) and an object $Y \in \mathbb{C}$, let $\varphi_{(X,c),Y}\colon \hom_{\mathbb{C}}(X, Y) \xrightarrow{\cong} \hom_{\mathbf{CoAlg}(F)}((X, c), GY)$ denote the $(X, c), Y)$th component of the natural bijection witnessing the adjunction $U \dashv G$.

Define a functor $P\colon \mathbf{CoAlg}(F \times H) \to \mathbf{CoAlg}(GHU)$ as follows:

1. for each $(F \times H)$-coalgebra $X \xrightarrow{c} FX \times HX$, we define

$$\begin{aligned} P(X, c) &\coloneqq \varphi_{(X,\pi_1^{FX\times HX}c),HX}\Big(X \xrightarrow{\pi_2^{FX\times HX}c} HX\Big) \\ &\qquad \in \hom_{\mathbf{CoAlg}(F)}((X, \pi_1^{FX\times HX}c),\, GHU(X, \pi_1^{FX\times HX}c)), \end{aligned}$$

   since $(X, \pi_1^{FX\times HX}c)$ is an F-coalgebra;

2. for each $(F \times H)$-coalgebra homomorphism $(X, c) \xrightarrow{f} (Y, d)$, we define $Pf \coloneqq f$. To see that f is a GHU-coalgebra homomorphism from $\big((X, \pi_1^{FX\times HX}c), P(X, c)\big)$ to $\big((Y, \pi_1^{FX\times HX}d), P(Y, d)\big)$, we perform the following calculation:

$$\begin{aligned} GHUf \circ P(X, c) &= GHUf \circ \varphi_{(X,\pi_1^{FX\times HX}c),HX}\Big(\pi_2^{FX\times HX}c\Big) \\ &= \varphi_{(X,\pi_1^{FX\times HX}c),HY}\Big(HUf \circ \pi_2^{FX\times HX} \circ c\Big) \\ &= \varphi_{(X,\pi_1^{FX\times HX}c),HY}\Big(\pi_2^{FY\times HY} \circ d \circ Uf\Big) \end{aligned}$$

$$
\begin{aligned}
&= \varphi_{(Y,\pi_1^{FY\times HY}d),HY}\left(\pi_2^{FY\times Hy}d\right)\circ f\\
&= P(Y,d)\circ f.
\end{aligned}
$$

In other words, the commuting diagram

$$
\begin{tikzcd}
\hom_{\mathbb{C}}(X,HX) \arrow[r, "\varphi_{(X,\pi_1^{FX\times HX}c),HX}", "\cong"'] \arrow[d, "HUf\circ(-)"'] & \hom_{\mathbf{CoAlg}(F)}((X,\pi_1^{FX\times HX}c),\, GHX) \arrow[d, "GHUf\circ(-)"] \\
\hom_{\mathbb{C}}(X,HY) \arrow[r, "\varphi_{(X,\pi_1^{FX\times HX}c),HY}", "\cong"'] & \hom_{\mathbf{CoAlg}(F)}((X,\pi_1^{FX\times HX}c),\, GHY) \\
\hom_{\mathbb{C}}(Y,HY) \arrow[r, "\cong", "\varphi_{(Y,\pi_1^{FY\times HY}d),HY}"'] \arrow[u, "(-)\circ Uf"] & \hom_{\mathbf{CoAlg}(F)}((Y,\pi_1^{FY\times HY}d),\, GHY) \arrow[u, "(-)\circ f"']
\end{tikzcd}
$$

gives rise to the chase of elements

$$
\begin{tikzcd}
\pi_2^{FX\times HX}c \arrow[r, mapsto, "\varphi_{(X,\pi_1^{FX\times HX}c),HX}"] \arrow[d, mapsto, "HUf\circ(-)"'] & P(X,c) \arrow[d, mapsto, "GHUf\circ(-)"] \\
Hf\circ\pi_2^{FX\times HX}\circ c = \pi_2^{FY\times HY}\circ d\circ f \arrow[r, mapsto, "\varphi_{(X,\pi_1^{FX\times HX}c),HY}"] & {} \\
\pi_2^{FY\times HY}d \arrow[r, mapsto, "\varphi_{(Y,\pi_1^{FY\times HY}d),HY}"'] \arrow[u, mapsto, "(-)\circ Uf"] & P(Y,d) \arrow[u, mapsto, "(-)\circ f"']
\end{tikzcd}
$$

where we obtained the equality $Hf\circ\pi_2^{FX\times HX}\circ c = \pi_2^{FY\times HY}\circ d\circ f$ from the fact that $d\circ f = (Ff\times Hf)\circ c$.

We need to show that this functor $P\colon \mathbf{CoAlg}(F\times H)\to\mathbf{CoAlg}(GHU)$ is an isomorphism of categories. To do this, we define a (suggestively named) functor $P^{-1}\colon \mathbf{CoAlg}(GHU)\to\mathbf{CoAlg}(F\times H)$ as follows:

1. for a GHU-coalgebra $((X,c),\alpha)$, where (X,c) is an F-coalgebra, let

   $$P^{-1}((X,c),\alpha) \coloneqq \langle c, \varphi^{-1}_{(X,c),HX}(\alpha)\rangle \in \hom_{\mathbb{C}}(X, FX\times HX),$$

   noting that $\varphi^{-1}_{(X,c),HX}(\alpha)\in\hom_{\mathbb{C}}(X,HX)$;

2. for a GHU-coalgebra homomorphism $((X,c),\alpha)\xrightarrow{f}((Y,d),\beta)$, let $P^{-1}f\coloneqq f$. The fact that f is an $(F\times H)$-coalgebra homomorphism from $(X,P^{-1}((X,c),\alpha))$ to $(Y,P^{-1}((Y,d),\beta))$ follows from the fact that f has to be an F-coalgebra homomorphism from (X,c) to (Y,d) as well as the commutativity of the diagram

$$
\begin{tikzcd}
\hom_{\mathbb{C}}(X,HX) \arrow[d, "HUf\circ(-)"'] & \hom_{\mathbf{CoAlg}(F)}((X,c),GHX) \arrow[l, "\varphi^{-1}_{(X,c),HX}"', "\cong"] \arrow[d, "GHUf\circ(-)"] \\
\hom_{\mathbb{C}}(X,HY) & \hom_{\mathbf{CoAlg}(F)}((X,c),GHY) \arrow[l, "\varphi^{-1}_{(X,c),HY}"', "\cong"] \\
\hom_{\mathbb{C}}(Y,HY) \arrow[u, "(-)\circ Uf"] & \hom_{\mathbf{CoAlg}(F)}((Y,d),GHY) \arrow[l, "\cong"', "\varphi^{-1}_{(Y,d),HY}"] \arrow[u, "(-)\circ f"']
\end{tikzcd}
$$

which would establish that $Hf \circ P^{-1}((X,c),\alpha) = P^{-1}((Y,d),\beta) \circ f$, in an argument similar to the proof that P really did send $(F \times H)$-coalgebra homomorphisms to GHU-coalgebra homomorphisms.

It is clear from the definitions of P and P^{-1} that $P^{-1} \circ P = \mathrm{id}_{\mathbf{CoAlg}(F\times H)}$ and $P \circ P^{-1} = \mathrm{id}_{\mathbf{CoAlg}(GHU)}$. Indeed, for any $(F \times H)$-coalgebra (X,c), we have

$$\begin{aligned}
P^{-1}(P(X,c)) &= P^{-1}\left((X, \pi_1^{FX\times HX} c),\ \varphi_{(X,\pi_1^{FX\times HX} c),HX}\left(\pi_2^{FX\times HX} c\right)\right) \\
&= \left(X,\ \left\langle \pi_1^{FX\times HX} c,\ \varphi^{-1}_{(X,\pi_1^{FX\times HX} c),HX}\left(\varphi_{(X,\pi_1^{FX\times HX} c),HX}\left(\pi_2^{FX\times HX} c\right)\right)\right\rangle\right) \\
&= (X, \langle \pi_1^{FX\times HX} c, \pi_2^{FX\times HX} c) \\
&= (X,c),
\end{aligned}$$

and for any GHU-coalgebra $((X,c),\alpha)$, we have

$$\begin{aligned}
P(P^{-1}((X,c),\alpha)) &= P(X,\ \langle c, \varphi^{-1}_{(X,c),HX}(\alpha)\rangle) \\
&= ((X,c),\ \varphi_{(X,c),HX}(\varphi^{-1}_{(X,c),HX}(\alpha))) \\
&= ((X,c),\alpha).
\end{aligned}$$

That $P^{-1} \circ P$ and $P \circ P^{-1}$ act as identity maps on morphisms in $\mathbf{CoAlg}(F \times H)$ and $\mathbf{CoAlg}(GHU)$ respectively follows immediately from the definitions of P and P^{-1} on morphisms. $\square$

Exercise 2.5.12

Consider two adjoint endofunctors as in

$$\begin{tikzcd} \mathbb{C} \arrow[loop left, "F"] \arrow[loop right, "G"] \end{tikzcd} \qquad \textit{with} \qquad F \dashv G.$$

Prove that we then get an isomorphism of categories

$$\mathbf{Alg}(F) \xrightarrow{\ \cong\ } \mathbf{CoAlg}(G)$$

between the associated categories of algebras and coalgebras.
(Remark: As noted in Arbib and Manes (1975a, theorem 5.7), when $\mathbb{C} = \mathbf{Sets}$ the only such adjunctions $F \dashv G$ are the product-exponent adjunctions $X \times (-) \dashv (-)^X$ as in (2.11). The argument goes as follows. In $\mathbf{Sets}$, each object A can be written as a coproduct $\coprod_{a\in A} 1$ of singletons. A left adjoint F must preserve such coproducts, so that $F(A) \cong \coprod_{a\in A} F(1) \cong F(1) \times A$. But then $G(-) \cong F(1) \Rightarrow (-)$, by uniqueness of adjoints.)

Solution. For $X, Y \in \mathbb{C}$, let $\varphi_{X,Y} \colon \hom_{\mathbb{C}}(FX,Y) \xrightarrow{\cong} \hom_{\mathbb{C}}(X,GY)$ denote the (X,Y)th component of the natural bijection witnessing the adjunction $F \dashv G$.

Define (suggestively named) functors $I \colon \mathbf{Alg}(F) \to \mathbf{CoAlg}(G)$ and $I^{-1} \colon \mathbf{CoAlg}(G) \to \mathbf{Alg}(F)$ as follows:

1. for an F-algebra $FX \xrightarrow{\alpha} X$, define $I(X,\alpha) \coloneqq \varphi_{X,X}(\alpha) \in \hom_{\mathbb{C}}(X,GX)$;

2. for an F-algebra homomorphism $(X,\alpha) \xrightarrow{f} (Y,\beta)$, define $If \coloneqq f$;

3. for a G-coalgebra $X \xrightarrow{c} GX$, define $I^{-1}(X, c) \coloneqq \varphi_{X,X}^{-1}(c) \in \hom_{\mathbb{C}}(FX, X)$;

4. for a G-coalgebra homomorphism $(X, c) \xrightarrow{f} (Y, d)$, define $I^{-1}f \coloneqq f$.

We ought to check that I (resp. I^{-1}) sends F-algebra homomorphisms to G-coalgebra homomorphisms (resp. G-coalgebra homomorphisms to F-algebra homomorphisms). We proceed similarly as in our solution to Exercise 2.5.11. Fix an F-algebra homomorphism $(X, \alpha) \xrightarrow{f} (Y, \beta)$. The commutativity of the diagram on the left

$$\begin{tikzcd}
\hom_{\mathbb{C}}(FX, X) \arrow[r, "\varphi_{X,X}", "\cong"'] \arrow[d, "f \circ (-)"'] & \hom_{\mathbb{C}}(X, GX) \arrow[d, "Gf \circ (-)"] \\
\hom_{\mathbb{C}}(FX, Y) \arrow[r, "\varphi_{X,Y}", "\cong"'] & \hom_{\mathbb{C}}(X, GY) \\
\hom_{\mathbb{C}}(FY, Y) \arrow[r, "\cong", "\varphi_{Y,Y}"'] \arrow[u, "(-) \circ Ff"] & \hom_{\mathbb{C}}(Y, GY) \arrow[u, "(-) \circ f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha \arrow[r, mapsto, "\varphi_{X,X}"] \arrow[d, mapsto, "f \circ (-)"'] & I(X, \alpha) \arrow[d, mapsto, "Gf \circ (-)"] \\
f\alpha = \beta F f \arrow[r, mapsto, "\varphi_{X,Y}"] & {} \\
\beta \arrow[r, mapsto, "\varphi_{Y,Y}"'] \arrow[u, mapsto, "(-) \circ Ff"] & I(Y, \beta) \arrow[u, mapsto, "(-) \circ f"']
\end{tikzcd}$$

gives rise to the chase of elements on the right, showing that f is a homomorphism of G-coalgebras from $(X, \varphi_{X,X}(\alpha))$ to $(Y, \varphi_{Y,Y}(\beta))$. A similar argument shows that $I^{-1}g$ is an F-algebra homomorphism whenever g is a G-coalgebra homomorphism.

Clearly, $I^{-1} \circ I = \mathrm{id}_{\mathbf{Alg}(F)}$ and $I \circ I^{-1} = \mathrm{id}_{\mathbf{CoAlg}(F)}$. □

Exercise 2.5.13

Theorem 2.5.9 deals with lifting adjunctions to categories of algebras. Check that its 'dual' version for coalgebras is (see Hermida and Jacobs (1998)) the following:

For functors $T\colon \mathbb{B} \to \mathbb{B}$, $G\colon \mathbb{A} \to \mathbb{B}$, $S\colon \mathbb{A} \to \mathbb{A}$, a natural transformation $\alpha\colon GS \Rightarrow TG$ induces a functor $\mathbf{CoAlg}(G)\colon \mathbf{CoAlg}(S) \to \mathbf{CoAlg}(T)$. Furthermore, if α is an isomorphism, then a left adjoint $F \dashv G$ induces a left adjoint $\mathbf{CoAlg}(F) \dashv \mathbf{CoAlg}(G)$. Does it require a new proof?

Solution. Recall that Theorem 2.5.9 asserts (with more detail) the following:

> For functors $P\colon \mathbb{C} \to \mathbb{C}$, $M\colon \mathbb{C} \to \mathbb{D}$, and $Q\colon \mathbb{D} \to \mathbb{D}$, and a natural transformation $\eta\colon QM \Rightarrow MP$,
>
> $$\begin{tikzcd}
\mathbb{D} \arrow[r, "Q"] \arrow[dr, Rightarrow, "\eta"] & \mathbb{D} \\
\mathbb{C} \arrow[r, "P"'] \arrow[u, "M"] & \mathbb{C} \arrow[u, "M"']
\end{tikzcd}$$
>
> we get an induced functor $\mathbf{Alg}(M)\colon \mathbf{Alg}(P) \to \mathbf{Alg}(Q)$. Furthermore, if M has a right adjoint, then $\mathbf{Alg}(M)$ also has a right adjoint.

In our situation, we have the following diagram:

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "T"] & \mathbb{B} \\
\mathbb{A} \arrow[r, "S"'] \arrow[u, "G"] & \mathbb{A} \arrow[u, "G"'] \arrow[ul, Rightarrow, "\alpha"']
\end{tikzcd}$$

Note that the natural transformation $\alpha\colon GS \Rightarrow TG$ yields a natural transformation $\alpha^{\mathrm{op}}\colon T^{\mathrm{op}}G^{\mathrm{op}} \Rightarrow G^{\mathrm{op}}S^{\mathrm{op}}$. So we have the diagram

$$\begin{tikzcd}
\mathbb{B}^{\mathrm{op}} \arrow[r, "T^{\mathrm{op}}"] \arrow[dr, Rightarrow, "\alpha^{\mathrm{op}}"] & \mathbb{B}^{\mathrm{op}} \\
\mathbb{A}^{\mathrm{op}} \arrow[r, "S^{\mathrm{op}}"'] \arrow[u, "G^{\mathrm{op}}"] & \mathbb{A}^{\mathrm{op}} \arrow[u, "G^{\mathrm{op}}"']
\end{tikzcd}$$

giving us an induced functor $\mathbf{Alg}(G^{\mathrm{op}})\colon \mathbf{Alg}(S^{\mathrm{op}}) \to \mathbf{Alg}(T^{\mathrm{op}})$. Since $\mathbf{Alg}(S^{\mathrm{op}}) = \mathbf{CoAlg}(S)^{\mathrm{op}}$ and $\mathbf{Alg}(T^{\mathrm{op}}) = \mathbf{CoAlg}(T)^{\mathrm{op}}$, we get an induced functor $\mathbf{CoAlg}(G)^{\mathrm{op}}\colon \mathbf{CoAlg}(S)^{\mathrm{op}} \to \mathbf{CoAlg}(T)^{\mathrm{op}}$, or equivalently, an induced functor $\mathbf{CoAlg}(G)\colon \mathbf{CoAlg}(S) \to \mathbf{CoAlg}(T)$. Now, if α is an isomorphism, then a right adjoint to G^{op} induces a right adjoint to $\mathbf{CoAlg}(G)^{\mathrm{op}}$. This can equivalently be phrased as: a left adjoint to G induces a left adjoint to $\mathbf{CoAlg}(G)$. $\square$

Exercise 2.5.14

A deterministic automaton $\langle \delta, \varepsilon\rangle\colon X \to X^A \times B$ *is called* ***observable*** *if its behaviour function* $\mathrm{beh} = \lambda x.\lambda\sigma.\varepsilon(\delta^*(x,\sigma))\colon X \to B^{A^\star}$ *from Proposition 2.3.5 is injective. Later, in Corollary 3.4.3, we shall see that this means that bisimilar states are equal.*

If this automaton comes equipped with an initial state $x_0 \in X$ *one calls the automaton* ***reachable*** *if the function* $\delta^*(x_0, -)\colon A^\star \to X$ *from Section 2.2 is surjective. This means that every state can be reached from the initial state* x_0 *via a suitable sequence of inputs. The automaton is called* ***minimal*** *if it is both observable and reachable.*

The realisation construction $\mathcal{R}\colon \mathbf{DB} \to \mathbf{DA}$ *from Proposition 2.5.8 clearly yields an observable automaton since the resulting behaviour function is the identity. An alternative construction, the* ***Nerode realisation****, gives a minimal automaton. It is obtained from a behaviour* $\psi\colon C^\star \to D$ *as follows. Consider the equivalence relation* $\equiv_\psi \subseteq C^\star \times C^\star$ *defined by*

$$\sigma \equiv_\psi \sigma' \iff \forall \tau \in C^\star.\psi(\sigma\cdot\tau) = \psi(\sigma'\cdot\tau).$$

We take the quotient $C^\star/\equiv_\psi$ *as state space; it is defined as the factorisation*

$$\begin{tikzcd} C^\star \arrow[rr, "\sigma\mapsto\lambda\tau.\psi(\sigma\cdot\tau)"] \arrow[dr, two heads] & & D^{C^\star} \\ & C^\star/\equiv_\psi \arrow[ur, tail] & \end{tikzcd}$$

This quotient carries an automaton structure with transition function $\delta_\psi\colon (C^\star/\equiv_\psi) \to (C^\star/\equiv_\psi)^C$ *given by* $\delta_\psi([\sigma])(c) = [\sigma\cdot c]$*, observation function* $\varepsilon_\psi\colon (C^\star/\equiv_\psi) \to D$ *defined as* $\varepsilon_\psi([\sigma]) = \psi(\sigma)$*, and initial state* $[\langle\rangle] \in C^\star/\equiv_\psi$*.*

1. *Check that this Nerode realisation* $\mathcal{N}(C^\star \xrightarrow{\psi} D)$ *is indeed a minimal automaton, forming a subautomaton/subcoalgebra* $C^\star/\equiv_\psi \rightarrowtail D^{C^\star}$ *of the final coalgebra.*

Write $\mathbf{RDA}$ *for the 'subcategory' of* $\mathbf{DA}$ *with reachable automata as objects, and morphisms* (f, g, h) *as in* $\mathbf{DA}$ *but with* f *as surjective function between the input sets. Similarly, let* $\mathbf{RDB}$ *be the subcategory of* $\mathbf{DB}$ *with the same objects but with morphisms* (f, g) *where* f *is surjective.*

2. *Check that the behaviour functor* $\mathcal{B}\colon \mathbf{DA} \to \mathbf{DB}$ *from Proposition 2.5.8 restricts to a functor* $\mathcal{B}\colon \mathbf{RDA} \to \mathbf{RDB}$*, and show that the Nerode realisation* $\mathcal{N}$ *yields a functor* $\mathbf{RDB} \to \mathbf{RDA}$ *in the opposite direction.*

3. *Prove that there is an adjunction* $\mathcal{B} \dashv \mathcal{N}$*.*

4. *Let* $\mathbf{MDA}$ *be the 'subcategory' of* $\mathbf{RDA}$ *with minimal automata as objects. Check that the adjunction in the previous point restricts to an equivalence of categories* $\mathbf{MDA} \simeq \mathbf{RDB}$*. Thus, (states of) minimal automata are in fact (elements of) final coalgebras.*

   (This result comes from Goguen (1972) and Goguen (1975); see also Adámek (1981).)

Solution. For a deterministic behaviour $\psi\colon C^\star \to D$ (i.e. an object in **DB**) and a word $\sigma \in C^\star$, we shall write $[\sigma]_\psi$ for the $\equiv_\psi$-equivalence class of σ. For a deterministic automaton $c\colon X \to X^A \times B$ (without a specified initial state), we let $\mathrm{beh}_c\colon X \to B^{A^\star}$ denote its behaviour function.

1. Let $\psi\colon C^\star \to D$ be a deterministic behaviour, i.e. an object in **DB**. We need to show that its Nerode realisation $\mathcal{N}\Big(C^* \xrightarrow{\psi} D\Big) \coloneqq \Big\langle C^\star/{\equiv_\psi} \xrightarrow{\langle\delta_\psi,\varepsilon_\psi\rangle} (C^\star/{\equiv_\psi})^C \times D,\ [\langle\rangle]_\psi \in C^\star/{\equiv_\psi}\Big\rangle$ is minimal.

   Reachability is easy: for any $\sigma \in C^\star$, we have $\delta_\psi([\langle\rangle]_\psi)(\sigma) = [\langle\rangle \cdot \sigma]_\psi = [\sigma]_\psi$. For observability, we first note that an induction on $\tau \in C^\star$ shows that

   $$\delta^*_\psi([\sigma]_\psi, \tau) = \delta_\psi([\sigma]_\psi)(\tau) = [\sigma \cdot \tau]_\psi \quad \text{for all } \sigma, \tau \in C^\star.$$

   Now suppose we are given a pair of words $\sigma, \sigma' \in C^\star$ satisfying the equality $\mathrm{beh}_{\langle\delta_\psi,\varepsilon_\psi\rangle}([\sigma]_\psi) = \mathrm{beh}_{\langle\delta_\psi,\varepsilon_\psi\rangle}([\sigma']_\psi)$. This implies that

   $$\varepsilon_\psi(\delta^*_\psi([\sigma]_\psi, \tau)) = \varepsilon_\psi(\delta^*_\psi([\sigma']_\psi, \tau)) \quad \text{for all } \tau \in C^\star,$$

   and hence

   $$\psi(\sigma \cdot \tau) = \psi(\sigma' \cdot \tau) \quad \text{for all } \tau \in C^\star,$$

   which gives us $[\sigma]_\psi = [\sigma']_\psi$.

2. Recall that the behaviour functor $\mathcal{B}\colon \mathbf{DA} \to \mathbf{DB}$ is defined on objects by sending a deterministic automaton $\langle X \xrightarrow{c} X^A \times B,\ x_0 \in X\rangle \in \mathbf{DA}$ to the deterministic behaviour $(A^\star \xrightarrow{\mathrm{beh}_c(x_0)} B) \in \mathbf{DB}$. On morphisms, $\mathcal{B}$ sends a morphism $\langle X \xrightarrow{\langle\delta,\varepsilon\rangle} X^A \times B,\ x_0 \in X\rangle \xrightarrow{(f,g,h)} \langle Y \xrightarrow{\langle\delta',\varepsilon'\rangle} Y^C \times D,\ y_0 \in Y\rangle$, where $A \xleftarrow{f} C$, $B \xrightarrow{g} D$, and $X \xrightarrow{h} Y$ are morphisms in **Sets** satisfying properties specified in Definition 2.5.6, to the morphism $(A^\star \xrightarrow{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}} B) \xrightarrow{(f,g)} (C^\star \xrightarrow{\mathrm{beh}_{\langle\delta',\varepsilon'\rangle}} D)$ in **DB**.

   Thus, if (f, g, h) is a morphism in **DA** with f surjective, then $\mathcal{B}(f, g, h) = (f, g)$ will be a morphism in **RDB**. So $\mathcal{B}$ restricts to a functor from **RDA** to **RDB**.

   Now, we already know that $\mathcal{N}$ sends a deterministic behaviour to a minimal automaton, and thus a reachable deterministic automaton. It remains to check that $\mathcal{N}$ extends to a functor from **RDB** to **RDA**. That is, we need to define $\mathcal{N}$ on morphisms in **RDB**. Suppose we are given two deterministic behaviours $A^\star \xrightarrow{\varphi} B$ and $C^\star \xrightarrow{\psi} D$, and a morphism $(A^\star \xrightarrow{\varphi} B) \xrightarrow{(f,g)} (C^\star \xrightarrow{\psi} D)$ in **RDB**, so that $f\colon C \to A$ is surjective and the diagram

   $$\begin{tikzcd} B \arrow[r, "g"] & D \\ A^\star \arrow[u, "\varphi"] & C^\star \arrow[l, "f^\star"] \arrow[u, "\psi"'] \end{tikzcd}$$

   in **Sets** commutes. Define $N(f, g)\colon A^\star/{\equiv_\varphi} \to C^\star/{\equiv_\psi}$ by

   $$N(f, g)([\sigma]_\varphi) \coloneqq [\tau]_\psi \quad \text{for some } \tau \in C^\star \text{ with } f^\star(\tau) = \sigma,$$

   for all $\sigma \in A^\star$. Such a τ exists for each σ because f is surjective. Furthermore, N is well-defined because if $\tau, \tau' \in C^\star$ are such that $f^\star(\tau) = f^\star(\tau')$, then for any $\rho \in C^\star$, we have

   $$\psi(\tau \cdot \rho) = (g\varphi f^\star)(\tau \cdot \rho) = (g\varphi f^\star)(\tau' \cdot \rho) = \psi(\tau \cdot \rho).$$

With this, we define $\mathcal{N}(f,g) := (f,g,N(f,g))$. Then f is surjective, $N(f,g)([\langle\rangle]_\varphi) = [\langle\rangle]_\psi$, and the diagram

$$\begin{tikzcd}
(A^\star/{\equiv_\varphi})^C \times D \arrow[rr, "(N(f,g))^C\times \mathrm{id}_D"] & & (C^\star/{\equiv_\psi})^C \times D \\
(A^\star/{\equiv_\varphi})^A \times B \arrow[u, "\mathrm{id}^f_{A^\star/\equiv_\varphi}\times g"] & & \\
A^\star/{\equiv_\varphi} \arrow[u, "\langle\delta_\varphi,\varepsilon_\varphi\rangle"] \arrow[rr, "N(f,g)"'] & & C^\star/{\equiv_\psi} \arrow[uu, "\langle\delta_\psi,\varepsilon_\psi\rangle"']
\end{tikzcd}$$

in **Sets** commutes. So $\mathcal{N}(f,g) = (f,g,N(f,g))$ is indeed a morphism in **RDA**. The functoriality of $\mathcal{N}$ follows quickly from definitions.

3. We will now establish the adjunction $\begin{tikzcd} \mathbf{RDA} \arrow[r, bend left, "\mathcal{B}"] \arrow[r, phantom, "\perp"] & \mathbf{RDB} \arrow[l, bend left, "\mathcal{N}"] \end{tikzcd}$ using Exercise 2.5.7. Define natural transformations $\eta\colon \mathrm{id}_{\mathbf{RDA}} \Rightarrow \mathcal{N}\mathcal{B}$ and $\theta\colon \mathcal{B}\mathcal{N} \Rightarrow \mathrm{id}_{\mathbf{RDB}}$ as follows.

For each reachable deterministic automaton $\mathbb{X} = \langle X \xrightarrow{\langle\delta,\varepsilon\rangle} X^A \times B,\ x_0 \in X\rangle$, define the triple $\eta_{\mathbb{X}} := (\mathrm{id}_A, \mathrm{id}_B, \eta^{\text{state}}_{\mathbb{X}})$ where $\eta^{\text{state}}_{\mathbb{X}}\colon X \to A^\star/{\equiv_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}}$ is given by

$$\eta^{\text{state}}_{\mathbb{X}}(x) := [\sigma]_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)} \quad \text{for some } \sigma \in A^\star \text{ with } \delta^*(x_0,\sigma) = x,$$

for all $x \in X$, exploiting the fact that is a $\mathbb{X}$ is a reachable automaton. This $\eta^{\text{state}}_{\mathbb{X}}$ is well-defined since if $\sigma, \sigma' \in A^\star$ are such that $\delta^*(x_0,\sigma) = \delta^*(x_0,\sigma')$, then for any $\tau \in A^\star$ we have

$$\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)(\sigma\cdot\tau) = \varepsilon(\delta^*(x_0,\sigma\cdot\tau)) = \varepsilon(\delta^*(x_0,\sigma'\cdot\tau)) = \mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)(\sigma'\cdot\tau)$$

and thus $[\sigma]_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)} = [\sigma']_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}$. This $\eta_{\mathbb{X}} = (\mathrm{id}_A, \mathrm{id}_B, \eta^{\text{state}}_{\mathbb{X}})$ is indeed a morphism in **RDA** because of the commutativity of the diagram

$$\begin{tikzcd}
X^A \times B \arrow[rr, "\left(\eta^{\text{state}}_{\mathbb{X}}\right)^A \times \mathrm{id}_B"] & & (A^\star/{\equiv_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}})^A \times B \\
X^A \times B \arrow[u, "\mathrm{id}^A_X \times \mathrm{id}_B"] & & \\
X \arrow[u, "\langle\delta,\varepsilon\rangle"] \arrow[rr, "\eta^{\text{state}}_{\mathbb{X}}"'] & & A^\star/{\equiv_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}} \arrow[uu, "\left\langle \delta_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}, \varepsilon_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)} \right\rangle"']
\end{tikzcd}$$

in **Sets** as well as the fact that $\eta^{\text{state}}_{\mathbb{X}}(x_0) = [\langle\rangle]_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}$. The naturality of η then quickly follows from the fact that the diagram

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\eta^{\text{state}}_X"'] & Y \arrow[d, "\eta^{\text{state}}_Y"] \\
A^\star/{\equiv_{\mathrm{beh}_c(x_0)}} \arrow[r, "N(f,g)"'] & C^\star/{\equiv_{\mathrm{beh}_d(y_0)}}
\end{tikzcd}$$

in **Sets** commutes whenever we have a morphism $\mathbb{X} = \langle X \xrightarrow{c} X^A \times B,\ x_0 \in X\rangle \xrightarrow{(f,g,h)} \mathbb{Y} = \langle Y \xrightarrow{d} Y^C \times D,\ y_0 \in Y\rangle$ in **RDA**.

For each deterministic behaviour $A^\star \xrightarrow{\varphi} B$, define the pair $\theta_\varphi \coloneqq (\mathrm{id}_A, \mathrm{id}_B)$, which is clearly a morphism in **RDB** from $\mathcal{BN}(A^\star \xrightarrow{\varphi} B)$ to $(A^\star \xrightarrow{\varphi} B)$, since

$$\begin{aligned}
\mathcal{BN}(A^\star \xrightarrow{\varphi} B) &= (A^\star \xrightarrow{\mathrm{beh}_{\langle \delta_\varphi, \varepsilon_\varphi\rangle}([\langle\rangle]_\varphi)} B) \\
&= (A^\star \xrightarrow{\lambda\sigma\in A^\star.\varepsilon_\varphi(\delta_\varphi^*([\langle\rangle]_\varphi,\sigma))} B) \\
&= (A^\star \xrightarrow{\lambda\sigma\in A^\star.\varepsilon_\varphi([\sigma]_\varphi)} B) \\
&= (A^\star \xrightarrow{\lambda\sigma\in A^\star.\varphi(\sigma)} B) \\
&= (A^\star \xrightarrow{\varphi} B).
\end{aligned}$$

As the components of θ are identity morphisms, it quickly follows that θ is indeed a natural transformation from $\mathcal{BN}$ to $\mathrm{id}_{\mathbf{RDB}}$. Note that θ is thus a natural isomorphism: for each deterministic behaviour $\varphi \in \mathbf{RDB}$, the morphism θ_φ is an isomorphism (in fact, it is an identity morphism) in **RDB**.

To establish the adjunction $\begin{tikzcd} \mathbf{RDA} \arrow[r, bend left, "\mathcal{B}"] \arrow[r, phantom, "\perp"] & \mathbf{RDB} \arrow[l, bend left, "\mathcal{N}"] \end{tikzcd}$, Exercise 2.5.7 tells us that it suffices to show the commutativity of the triangles

$$\begin{tikzcd}
\mathcal{B} \arrow[r, Rightarrow, "\mathcal{B}\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_{\mathcal{B}}"'] & \mathcal{BNB} \arrow[d, Rightarrow, "\theta_{\mathcal{B}}"] \\
& \mathcal{B}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{N} \arrow[r, Rightarrow, "\eta\mathcal{N}"] \arrow[dr, Rightarrow, "\mathrm{id}_{\mathcal{N}}"'] & \mathcal{NBN} \arrow[d, Rightarrow, "\mathcal{N}\theta"] \\
& \mathcal{N}
\end{tikzcd}$$

in the functor categories $[\mathbf{RDA}, \mathbf{RDB}]$ and $[\mathbf{RDB}, \mathbf{RDA}]$ respectively (where, for categories $\mathbb{C}$ and $\mathbb{D}$, the **functor category** $[\mathbb{C}, \mathbb{D}]$ has functors from $\mathbb{C}$ to $\mathbb{D}$ as objects and natural transformations between such functors as morphisms). This says that for any deterministic automaton $\mathbb{X}$ and any deterministic behaviour φ, the triangles

$$\begin{tikzcd}
\mathcal{B}\mathbb{X} \arrow[r, "\mathcal{B}\eta_{\mathbb{X}}"] \arrow[dr, "\mathrm{id}_{\mathcal{B}\mathbb{X}}"'] & \mathcal{BNB}\mathbb{X} \arrow[d, "\theta_{\mathcal{B}\mathbb{X}}"] \\
& \mathcal{B}\mathbb{X}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{N}\varphi \arrow[r, "\eta_{\mathcal{N}\varphi}"] \arrow[dr, "\mathrm{id}_{\mathcal{N}\varphi}"'] & \mathcal{NBN}\varphi \arrow[d, "\mathcal{N}\theta_\varphi"] \\
& \mathcal{N}\varphi
\end{tikzcd}$$

in **RDB** and **RDA** respectively commute. Indeed, writing $\mathbb{X} = \langle X \xrightarrow{c} X^A \times B,\ x_0 \in X\rangle$, the triangle $\begin{tikzcd}
\mathcal{B}\mathbb{X} \arrow[r, "\mathcal{B}\eta_{\mathbb{X}}"] \arrow[dr, "\mathrm{id}_{\mathcal{B}\mathbb{X}}"'] & \mathcal{BNB}\mathbb{X} \arrow[d, "\theta_{\mathcal{B}\mathbb{X}}"] \\
& \mathcal{B}\mathbb{X}
\end{tikzcd}$ in **RDB** is simply the triangle $\begin{tikzcd}
\mathrm{beh}_c(x_0) \arrow[r, "(\mathrm{id}_A{,}\mathrm{id}_B)"] \arrow[dr, "(\mathrm{id}_A{,}\mathrm{id}_B)"'] & \mathrm{beh}_c(x_0) \arrow[d, "(\mathrm{id}_A{,}\mathrm{id}_B)"] \\
& \mathrm{beh}_c(x_0)
\end{tikzcd}$

in **RDB**. Furthermore, writing $\varphi = (C^\star \xrightarrow{\varphi} D)$, the triangle
$$\begin{tikzcd} \mathcal{N}\varphi \arrow[r, "\eta_{\mathcal{N}\varphi}"] \arrow[dr, "\mathrm{id}_{\mathcal{N}\varphi}"'] & \mathcal{N}\mathcal{B}\mathcal{N}\varphi \arrow[d, "\mathcal{N}\theta_\varphi"] \\ & \mathcal{N}\varphi \end{tikzcd}$$
in **RDA**

is the triangle
$$\begin{tikzcd} \mathcal{N}\varphi \arrow[r, "{(\mathrm{id}_C,\mathrm{id}_D,\eta^{\mathrm{state}}_{\mathcal{N}\varphi})}"] \arrow[dr, "{(\mathrm{id}_C,\mathrm{id}_D,\mathrm{id}_{C^\star/\equiv_\varphi})}"'] & \mathcal{N}\varphi \arrow[d, "{(\mathrm{id}_C,\mathrm{id}_D,N(\mathrm{id}_C,\mathrm{id}_D)}"] \\ & \mathcal{N}\varphi \end{tikzcd}$$
in **RDA**, which commutes because $\big(N(\mathrm{id}_C, \mathrm{id}_D) \circ \eta^{\mathrm{state}}_{\mathcal{N}\varphi}\big)([\sigma]_\varphi) = N(\mathrm{id}_C, \mathrm{id}_D)([\sigma]_\varphi) = [\sigma]_\varphi$ for all $\sigma \in C^\star$.

4. As $\mathcal{N}$ will send any object in **RDB** to an object in **MDA** the adjunction $\begin{tikzcd} \mathbf{RDA} \arrow[r, bend left, "\mathcal{B}"] \arrow[r, phantom, "\perp"] & \mathbf{RDB} \arrow[l, bend left, "\mathcal{N}"] \end{tikzcd}$ restricts to an adjunction $\begin{tikzcd} \mathbf{MDA} \arrow[r, bend left, "\mathcal{B}"] \arrow[r, phantom, "\perp"] & \mathbf{RDB} \arrow[l, bend left, "\mathcal{N}"] \end{tikzcd}$ with the same unit η and counit θ. We have already observed that θ is a natural isomorphism. Let $\mathbb{X}$ be a minimal automaton. The injectivity of its behaviour function implies that $\eta^{\mathrm{state}}_{\mathbb{X}}$ is also injective (and $\eta^{\mathrm{state}}_{\mathbb{X}}$ is surjective by definition), so that $\eta^{\mathrm{state}}_{\mathbb{X}}$ is an isomorphism in **Sets**. Furthermore, writing $\mathbb{X} = (X \xrightarrow{\langle\delta,\varepsilon\rangle} X^A \times B,\ x_0 \in X)$, the commutativity of the diagram
$$\begin{tikzcd} X^A \times B \arrow[r, "{\left(\eta^{\mathrm{state}}_{\mathbb{X}}\right)^A \times \mathrm{id}_B}"] & (A^\star/{\equiv_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}})^A \times B \\ X^A \times B \arrow[u, "\mathrm{id}_X^A \times \mathrm{id}_B"] & \\ X \arrow[u, "{\langle\delta,\varepsilon\rangle}"] \arrow[r, "\eta^{\mathrm{state}}_{\mathbb{X}}"'] & A^\star/{\equiv_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}} \arrow[uu, "{\left\langle \delta_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}, \varepsilon_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)} \right\rangle}"'] \end{tikzcd}$$
in **Sets** together with the fact that $\eta^{\mathrm{state}}_{\mathbb{X}}(x_0) = [\langle\rangle]_{\mathrm{beh}_{\langle\delta,\varepsilon\rangle}(x_0)}$ implies that $\eta_{\mathbb{X}}$ is an isomorphism in **RDA**. Therefore the adjunction $\begin{tikzcd} \mathbf{MDA} \arrow[r, bend left, "\mathcal{B}"] \arrow[r, phantom, "\perp"] & \mathbf{RDB} \arrow[l, bend left, "\mathcal{N}"] \end{tikzcd}$ is in fact an adjoint equivalence. $\square$

Exercise 2.5.15

This exercise and the next one continue the description of linear dynamical systems from Exercise 2.2.12. Here we look at the duality between reachability and observability. First, a preliminary result.

1. *Let B be an arbitrary vector space. Prove that the final coalgebra in* **Vect** *of the functor $X \mapsto B \times X$ is the set of infinite sequences $B^{\mathbb{N}}$, with obvious vector space structure, and with coalgebra structure $\langle \mathsf{hd}, \mathsf{tl}\rangle \colon B^{\mathbb{N}} \xrightarrow{\cong} B \times B^{\mathbb{N}}$ given by head and tail.*

Call a linear dynamical system $A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B$ ***reachable*** *if the induced function $\mathrm{int}_{[F,G]} \colon A^\S \to$*

X in the diagram on the left below is surjective (or equivalently, an epimorphism in **Vect***).*

$$\begin{tikzcd}
A + A^{\S} \arrow[r, "\mathrm{id}_A + \mathrm{int}_{[F,G]}"] \arrow[d, "\cong"'] & A + X \arrow[d, "{[F,G]}"'] \\
A^{\S} \arrow[r, "\mathrm{int}_{[F,G]}"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times X \arrow[r, "\mathrm{id}_B \times \mathrm{beh}_{\langle H,G\rangle}"] & B \times B^{\mathbb{N}} \\
X \arrow[u, "{\langle H,G\rangle}"] \arrow[r, "\mathrm{beh}_{\langle H,G\rangle}"'] & B^{\mathbb{N}} \arrow[u, "\cong", "\langle \mathsf{hd},\mathsf{tl}\rangle"']
\end{tikzcd}$$

Similarly, call this system ***observable*** *if the map $\mathrm{beh}_{\langle H,G\rangle}\colon X \to B^{\mathbb{N}}$ on the right is injective (equivalently, a monomorphism in* **Vect***). And call the system* ***minimal*** *if it is both reachable and observable.*

2. *Prove that $A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B$ is reachable in* **Vect** *if and only if $B \xrightarrow{H} X \xrightarrow{G} X \xrightarrow{F} A$ is observable in* $\mathbf{Vect}^{\mathrm{op}}$.

(Kalman's duality result (Kálmán et al., 1969, chapter 2) is now an easy consequence in finite-dimensional vector spaces, where the adjoint operator $(-)^*$ makes **Vect** *isomorphic to* $\mathbf{Vect}^{\mathrm{op}}$ *— where V^* is of course the 'dual' vector space of linear maps to the underlying field. This result says that $A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B$ is reachable if and only if $B^* \xrightarrow{H^*} X^* \xrightarrow{G^*} X^* \xrightarrow{F^*} A^*$ is observable. See also Arbib and Manes (1975a). There is also a duality result for bialgebras in Bidoit, Hennicker, and Kurz (2001).)*

Solution.

1. For a $(B \times (-))$-coalgebra $X \xrightarrow{\langle H,G\rangle} B \times X$ and $x_0 \in X$, letting $(b_n, x_{n+1}) \coloneqq \langle H,G\rangle(x_n)$ for all $n \in \mathbb{N}$, the behaviour map $\mathrm{beh}_{\langle H,G\rangle}$ must satisfy $\mathrm{beh}_{\langle H,G\rangle}(x_0) = (b_0, b_1, b_2, \ldots)$.

2. Recall that **Vect** consists of only the finite-dimensional vector spaces over $\mathbb{R}$. So products and coproducts coincide in **Vect**. So we obtain this result simply because $\mathbf{Vect}^{\mathrm{op}}$ is dual to **Vect**. □

Exercise 2.5.16
This exercise sketches an adjunction capturing Kalman's minimal realisation (Kálmán, Falb, and Arbib, 1969, chapter 10) for linear dynamical systems, in analogy with the Nerode realisation, described in Exercise 2.5.14. This categorical version is based on Arbib and Manes (1974) and Arbib and Manes (1975a).

Form the category **RLDS** *of reachable linear dynamical systems. Its objects are such reachable systems $A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B$ in* **Vect***. And its morphisms from $A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B$ in* **Vect** *to $C \xrightarrow{F'} Y \xrightarrow{G'} Y \xrightarrow{H'} D$ are triples of functions $C \overset{f}{\twoheadrightarrow} A$, $B \xrightarrow{g} D$ and $X \xrightarrow{h} Y$ with*

$$\begin{tikzcd}
A \arrow[r, "F"] & X \arrow[r, "G"] \arrow[d, "h"] & X \arrow[r, "H"] \arrow[d, "h"] & B \arrow[d, "g"] \\
C \arrow[u, two heads, "f"] \arrow[r, "F'"'] & Y \arrow[r, "G'"'] & Y \arrow[r, "H'"'] & D
\end{tikzcd}$$

Note that f is required to be a surjection/epimorphism.

Also, there is a category **RLB** *with linear maps $\varphi\colon A^{\S} \to B$ as objects. A morphism $(A^{\S} \xrightarrow{\varphi} B) \to (C^{\S} \xrightarrow{\psi} D)$ is a pair of linear maps $f\colon C \twoheadrightarrow A$ and $g\colon B \to D$ with $g \circ \varphi \circ f^{\S} = \psi$ — where $f^{\S}$ results from the functoriality of $(-)^{\S}$; see Exercise 2.4.8.3.*

1. *Demonstrate that the behaviour formula from Exercise 2.2.12.4 and Exercise 2.4.8.2 yields a behaviour functor* $\mathcal{B}\colon \mathbf{RLDS} \to \mathbf{RLB}$, *given by* $(F, G, H) \mapsto H \circ \mathrm{int}_{[F,G]}$.

2. *Construct a functor* $\mathcal{K}\colon \mathbf{RLB} \to \mathbf{RLDS}$ *in the reverse direction in the following way. Assume a behaviour* $\psi\colon C^{\S} \to D$, *and form the behaviour map* $b = \mathrm{beh}_{\langle \psi, \mathsf{sh}\rangle}\colon C^{\S} \to D^{\mathbb{N}}$ *below, using the finality from the previous exercise:*

$$\begin{tikzcd} D \times C^{\S} \arrow[r, dashed, "\mathrm{id}_D \times b"] & D \times D^{\mathbb{N}} \\ C^{\S} \arrow[u, "\langle \psi, \mathsf{sh}\rangle"] \arrow[r, dashed, "b"'] & D^{\mathbb{N}} \arrow[u, "\cong"', "\langle \mathsf{hd}, \mathsf{tl}\rangle"] \end{tikzcd}$$

The image $\mathrm{Im}(b)$ *of this behaviour map can be written as*

$$\Big(C^{\S} \xrightarrow{\mathrm{beh}_{\langle \psi, \mathsf{sh}\rangle}} D^{\mathbb{N}}\Big) = \Big(C^{\S} \overset{e}{\twoheadrightarrow} \mathrm{Im}(b) \overset{m}{\rightarrowtail} D^{\mathbb{N}}\Big).$$

It is not hard to see that the tail function $\mathsf{tl}\colon B^{\mathbb{N}} \to B^{\mathbb{N}}$ *restricts to* $\mathsf{tl}'\colon \mathrm{Im}(b) \to \mathrm{Im}(b)$ *via diagonal fill-in:*

$$\begin{tikzcd} C^{\S} \arrow[r, two heads, "e"] \arrow[d, "e \circ \mathsf{sh}"'] & \mathrm{Im}(b) \arrow[d, "\mathsf{tl} \circ m"] \arrow[dl, dashed, "\mathsf{tl}'"] \\ \mathrm{Im}(b) \arrow[r, tail, "m"'] & D^{\mathbb{N}} \end{tikzcd}$$

Hence one can define a linear dynamical system as

$$\mathcal{K}\Big(C^{\S} \xrightarrow{\psi} D\Big) \overset{\mathrm{def}}{=} \Big(C \xrightarrow{e \circ \mathsf{in}} \mathrm{Im}(b) \xrightarrow{\mathsf{tl}'} \mathrm{Im}(b) \xrightarrow{\mathsf{hd} \circ m} D\Big).$$

Prove that this gives a minimal realisation in an adjunction $\mathcal{B} \dashv \mathcal{K}$.

Solution.

1. We need to define the behaviour functor $\mathcal{B}\colon \mathbf{RLDS} \to \mathbf{RLB}$ on morphisms. Suppose we have morphisms $C \overset{f}{\twoheadrightarrow} A$, $B \xrightarrow{g} D$, and $X \xrightarrow{h} Y$ in $\mathbf{Vect}$ which constitute a morphism from a reachable linear dynamical system $A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B$ to another reachable linear dynamical system $C \xrightarrow{F'} Y \xrightarrow{G'} Y \xrightarrow{H'} D$, so that the diagram

$$\begin{tikzcd} A \arrow[r, "F"] & X \arrow[r, "G"] \arrow[d, "h"] & X \arrow[r, "H"] \arrow[d, "h"] & B \arrow[d, "g"] \\ C \arrow[u, two heads, "f"] \arrow[r, "F'"'] & Y \arrow[r, "G'"'] & Y \arrow[r, "H'"'] & D \end{tikzcd}$$

in $\mathbf{Vect}$ commutes.

We first claim that the square

$$\begin{tikzcd} A^{\S} \arrow[r, "\mathrm{int}_{[F,G]}"] & X \arrow[d, "h"] \\ C^{\S} \arrow[u, "f^{\S}"] \arrow[r, "\mathrm{int}_{[F',G']}"'] & Y \end{tikzcd}$$

in **Vect** commutes. Indeed, for any $(c_0, c_1, c_2, \ldots, c_n, 0_C, 0_C, 0_C, \ldots) \in C^{\S}$, where 0_C is the zero vector in C, we have

$$\begin{aligned}
&\mathrm{int}_{[F',G']}(c_0, c_1, c_2, \ldots, c_n, 0_C, 0_C, 0_C, \ldots) \\
&= F'(c_0) +_Y (G'F')(c_1) +_Y ((G')^2F')(c_2) +_Y \cdots +_Y ((G')^nF')(c_n) \\
&= (hFf)(c_0) +_Y (G'hFf)(c_1) +_Y ((G')^2hFf)(c_2) +_Y \cdots +_Y ((G')^nhFf)(c_n) \\
&= (hFf)(c_0) +_Y (hGFf)(c_1) +_Y (hG^2Ff)(c_2) +_Y \cdots +_Y (hG^nFf)(c_n) \\
&= h\big(F(f(c_0)) +_X (GF)(f(c_1)) +_X (G^2F)(f(c_2)) +_X \cdots +_X (G^nF)(f(c_n))\big) \\
&= h\big(\mathrm{int}_{[F,G]}(f(c_0), f(c_1), f(c_2), \ldots, f(c_n), 0_C, 0_C, 0_C, \ldots)\big) \\
&= h\Big(\mathrm{int}_{[F,G]}\big(f^{\S}(c_0, c_1, c_2, \ldots, c_n, 0_C, 0_C, 0_C, \ldots)\big)\Big),
\end{aligned}$$

where $+_X$ and $+_Y$ are the vector additions in X and Y respectively.

Consequently, the diagram

$$\begin{tikzcd} A^{\S} \arrow[r, "\mathrm{int}_{[F,G]}"] & X \arrow[r, "H"] \arrow[d, "h"] & B \arrow[d, "g"] \\ C^{\S} \arrow[u, "f^{\S}"] \arrow[r, "\mathrm{int}_{[F',G']}"'] & Y \arrow[r, "H'"'] & D \end{tikzcd}$$

in **Vect** commutes, making (f, g) a morphism in **RLB** from $A^{\S} \xrightarrow{H \circ \mathrm{int}_{[F,G]}} B$ to $C^{\S} \xrightarrow{H' \circ \mathrm{int}_{[F',G']}} D$. So we define $\mathcal{B}(f, g, h) := (f, g)$.

2. Fix a linear map $C^{\S} \xrightarrow{\psi} D$, i.e. an object in **RLB**. Let $C + C^{\S} \xrightarrow[\cong]{[\mathsf{in}_C, \mathsf{sh}_C]} C^{\S}$ and $D^{\mathbb{N}} \xrightarrow[\cong]{\langle \mathsf{hd}_D, \mathsf{tl}_D \rangle} D \times D^{\mathbb{N}}$ respectively denote the initial $(C + (-))$-algebra and the final $(D \times (-))$-coalgebra, from Exercise 2.4.8.1 and Exercise 2.5.15.1. Let $C^{\S} \xrightarrow{b_\psi} D^{\mathbb{N}}$ be the unique linear map making the diagram

$$\begin{tikzcd} D \times C^{\S} \arrow[r, "\mathrm{id}_D \times b_\psi"] & D \times D^{\mathbb{N}} \\ C^{\S} \arrow[u, "\langle \psi, \mathsf{sh}_C \rangle"] \arrow[r, "b_\psi"'] & D^{\mathbb{N}} \arrow[u, "\cong"', "\langle \mathsf{hd}_D, \mathsf{tl}_D \rangle"] \end{tikzcd}$$

in **Vect** commute, and let $\mathrm{Im}(b_\psi)$ denote the linear subspace of $D^{\mathbb{N}}$ which is the image of b_ψ. Then define the linear maps $e_\psi \colon C^{\S} \to \mathrm{Im}(b_\psi)$ and $m_\psi \colon \mathrm{Im}(b_\psi) \to D^{\mathbb{N}}$ by

$$e_\psi(v) := b_\psi(v) \quad \text{and} \quad m_\psi(w) := w$$

for all $v \in C^{\S}$ and $w \in D^{\mathbb{N}}$. Then e_ψ and m_ψ are, respectively, an epimorphism and a monomorphism in **Vect**, and $b_\psi = m_\psi e_\psi$. Also define $\mathsf{tl}'_\psi \colon \mathrm{Im}(b_\psi) \to \mathrm{Im}(b_\psi)$ by

$$\mathsf{tl}'_\psi(w) := \mathsf{tl}_D(w)$$

for all $w \in \mathrm{Im}(b_\psi) \subseteq D^{\mathbb{N}}$, so that $\mathsf{tl}_D \circ m_\psi = m_\psi \circ \mathsf{tl}'_\psi$ and $\mathsf{tl}'_\psi \circ e_\psi = e_\psi \circ \mathsf{sh}_C$.

First we check that the linear dynamical system

$$\mathcal{K}\Big(C^{\S} \xrightarrow{\psi} D\Big) := \Big(C \xrightarrow{e_\psi \circ \mathsf{in}_C} \mathrm{Im}(b_\psi) \xrightarrow{\mathsf{tl}'_\psi} \mathrm{Im}(b_\psi) \xrightarrow{\mathsf{hd}_D \circ m_\psi} D\Big)$$

is indeed minimal. First, we show that it is reachable, i.e. the linear map $C^{\S} \xrightarrow{\mathrm{int}_{\left[e_\psi \circ \mathsf{in}_C, \mathsf{tl}'_\psi\right]}} \mathrm{Im}(b_\psi)$ making the diagram

$$\begin{tikzcd}
C + C^{\S} \arrow[r, "\mathrm{id}_C \,+\, \mathrm{int}_{\left[e_\psi \circ \mathsf{in}_C, \mathsf{tl}'_\psi\right]}"] \arrow[d, "{[\mathsf{in}_C, \mathsf{sh}_C]}"', "\cong"] & C + \mathrm{Im}(b_\psi) \arrow[d, "{\left[e_\psi \circ \mathsf{in}_C, \mathsf{tl}'_\psi\right]}"] \\
C^{\S} \arrow[r, "\mathrm{int}_{\left[e_\psi \circ \mathsf{in}_C, \mathsf{tl}'_\psi\right]}"'] & \mathrm{Im}(b_\psi)
\end{tikzcd}$$

in **Vect** commute is surjective. Indeed, for any $w \in \mathrm{Im}(b_\psi) \subseteq D^{\mathbb{N}}$, there exists $v \in C^{\S}$ such that $b_\psi(v) = w$, by definition of $\mathrm{Im}(b_\psi)$. So if we write, $v = (c_0, c_1, \dots, c_n, 0_C, 0_C, 0_C, \dots)$, then

$$\begin{aligned}
\mathrm{int}_{[e_\psi \circ \mathsf{in}_C, \mathsf{tl}'_\psi]}(v) &= e_\psi(\mathsf{in}_C(c_0)) +_D \mathsf{tl}'_\psi\big(\mathrm{int}_{[e_\psi \circ \mathsf{in}_C, \mathsf{tl}'_\psi]}(c_1, \dots, c_n, 0_C, 0_C, 0_C, \dots)\big) \\
&= (e_\psi \circ \mathsf{in}_C)(c_0) +_D (\mathsf{tl}'_\psi \circ e_\psi \circ \mathsf{in}_C)(c_1) +_D \cdots +_D ((\mathsf{tl}'_\psi)^n \circ e_\psi \circ \mathsf{in}_C)(c_n) \\
&= (e_\psi \circ \mathsf{in}_C)(c_0) +_D (e_\psi \circ \mathsf{sh}_C \circ \mathsf{in}_C)(c_1) +_D \cdots +_D (e_\psi \circ (\mathsf{sh}_C)^n \circ \mathsf{in}_C)(c_n) \\
&= e_\psi\big(\mathsf{in}_C(c_0) +_{C^{\S}} (\mathsf{sh}_C \circ \mathsf{in}_C)(c_1) +_{C^{\S}} \cdots +_{C^{\S}} ((\mathsf{sh}_C)^n \circ \mathsf{in}_C)(c_n)\big) \\
&= e_\psi(c_0, c_1, \dots, c_n, 0_C, 0_C, 0_C, \dots) \\
&= b_\psi(v) \\
&= w.
\end{aligned}$$

Thus $\mathcal{K}(\psi)$ is reachable. Now we show that the linear map $\mathrm{Im}(b_\psi) \xrightarrow{\mathrm{beh}_{\left\langle \mathsf{hd}_D \circ m_\psi, \mathsf{tl}'_\psi \right\rangle}} D^{\mathbb{N}}$ making the diagram

$$\begin{tikzcd}
D \times \mathrm{Im}(b_\psi) \arrow[r, "\mathrm{id}_D \,\times\, \mathrm{beh}_{\left\langle \mathsf{hd}_D \circ m_\psi, \mathsf{tl}'_\psi \right\rangle}"] & D \times D^{\mathbb{N}} \\
\mathrm{Im}(b_\psi) \arrow[u, "\left\langle \mathsf{hd}_D \circ m_\psi, \mathsf{tl}'_\psi \right\rangle"] \arrow[r, "\mathrm{beh}_{\left\langle \mathsf{hd}_D \circ m_\psi, \mathsf{tl}'_\psi \right\rangle}"'] & D^{\mathbb{N}} \arrow[u, "\cong", "\left\langle \mathsf{hd}_D, \mathsf{tl}_D \right\rangle"']
\end{tikzcd}$$

in **Vect** commute is injective, which would establish the observability of $\mathcal{K}(\psi)$. The injectivity of $\mathrm{beh}_{\left\langle \mathsf{hd}_D \circ m_\psi, \mathsf{tl}'_\psi \right\rangle}$ follows quickly from the fact that $\langle \mathsf{hd}_D \circ m_\psi, \mathsf{tl}'_\psi \rangle$ is simply $\langle \mathsf{hd}_D, \mathsf{tl}_D \rangle$ restricted to $\mathsf{Im}(b_\psi)$.

Next, we need to extend $\mathcal{K}$ to a functor from **RLB** to **RLDS** by defining it on the morphisms in **RLB**. Suppose we are given a morphism $(A^{\S} \xrightarrow{\varphi} B) \xrightarrow{(f,g)} (C^{\S} \xrightarrow{\psi} D)$ in **RLB**, so f is epic and we have the commuting square

$$\begin{tikzcd}
A^{\S} \arrow[r, "\varphi"] & B \arrow[d, "g"] \\
C^{\S} \arrow[u, "f^{\S}"] \arrow[r, "\psi"] & D
\end{tikzcd}$$

in **Vect**. Define $h_{\varphi,\psi,f,g} \colon \mathrm{Im}(b_\varphi) \to \mathrm{Im}(b_\psi)$ as follows. For $v = (a_0, \dots, a_n, 0_A, 0_A, 0_A, \dots) \in A^{\S}$, let $c_0, \dots, c_n \in C$ be such that $f(c_i) = a_i$ for each $i \in \{0, \dots, n\}$, noting that we exploited the surjectivity of f here, and write $w := (c_0, \dots, c_n, 0_C, 0_C, 0_C, \dots)$. Note that if $w' = (c'_0, \dots, c'_n, 0_C, 0_C, 0_C, \dots) \in C^{\S}$ is also such that $f(c'_i) = f(c_i) = a_i$ for all $i \in \{0, \dots, n\}$, then, for all $k \in \mathbb{N}$, we have

$$\psi(\mathsf{sh}^k_C(w)) = (g \circ \varphi \circ f^{\S})(\mathsf{sh}^k_C(w)) = (g \circ \varphi)(\mathsf{sh}^k_A(v)) = (g \circ \varphi \circ f^{\S})(\mathsf{sh}^k_C(w')) = \psi(\mathsf{sh}^k_C(w')),$$

and so $b_\psi(w) = b_\psi(w')$. Thus we can define

$$h_{\varphi,\psi,f,g}(b_\varphi(v)) := b_\psi(w).$$

Then the diagram

$$\begin{tikzcd}
A \arrow[r, "e_\varphi \circ \mathsf{in}_A"] & \mathrm{Im}(b_\varphi) \arrow[r, "\mathsf{tl}'_\varphi"] \arrow[d, "h_{\varphi,\psi,f,g}"] & \mathrm{Im}(b_\varphi) \arrow[r, "\mathsf{hd}_B \circ m_\varphi"] \arrow[d, "h_{\varphi,\psi,f,g}"] & B \arrow[d, "g"] \\
C \arrow[u, two heads, "f"] \arrow[r, "e_\psi \circ \mathsf{in}_C"'] & \mathrm{Im}(b_\psi) \arrow[r, "\mathsf{tl}'_\psi"'] & \mathrm{Im}(b_\psi) \arrow[r, "\mathsf{hd}_D \circ m_\psi"'] & D
\end{tikzcd}$$

in **Vect** commutes, making $(f, g, h_{\varphi,\psi,f,g})$ a morphism in **RLDS**. So we can define the functor $\mathcal{K} \colon \mathbf{RLB} \to \mathbf{RLDS}$ on morphisms by $\mathcal{K}(f,g) := \mathcal{K}(f, g, h_{\varphi,\psi,f,g})$.

Finally, we show that $\mathcal{B} \dashv \mathcal{K}$. Fix an object $(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B) \in \mathbf{RLDS}$ and an object $(C^{\S} \xrightarrow{\psi} D) \in \mathbf{RLB}$. We will show that a pair the (f,g) is a morphism $\mathcal{B}(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B) \xrightarrow{(f,g)} (C^{\S} \xrightarrow{\psi} D)$ in **RLB** if and only if there exists a unique linear map $h \colon X \to \mathrm{Im}(b_\psi)$ such that $(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B) \xrightarrow{(f,g,h)} \mathcal{K}(C^{\S} \xrightarrow{\psi} D)$ is a morphism in **RLDS**.

Suppose $\mathcal{B}(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B) \xrightarrow{(f,g)} (C^{\S} \xrightarrow{\psi} D)$ is a morphism in **RLB**, so that the diagram

$$\begin{tikzcd}
A + A^{\S} \arrow[r, "\mathrm{id}_A + \mathrm{int}_{[F,G]}"] \arrow[d, "{[\mathsf{in}_A, \mathsf{sh}_A]}"', "\cong"] & A + X \arrow[d, "{[F,G]}"] & \\
A^{\S} \arrow[r, "\mathrm{int}_{[F,G]}"'] & X \arrow[r, "H"'] & B \arrow[d, "g"] \\
C^{\S} \arrow[u, "f^{\S}"] \arrow[rr, "\psi"'] & & D
\end{tikzcd}$$

in **Vect** commutes. Define $h\colon X \to D^{\mathbb{N}}$ by

$$h(x) \coloneqq \Big((gHG^k)(x)\Big)_{k\in\mathbb{N}}$$

for all $x \in X$. For any $x \in X$, as the linear dynamical system $(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B)$ is reachable, there exists $v = (a_0, \dots, a_n, 0_A, 0_A, 0_A, \dots) \in A^{\S}$ such that $\mathrm{int}_{[F,G]}(v) = \sum_{j=0}^{n} (G^j F)(a_j) = x$. Furthermore, as $f\colon C \to A$ is surjective there exist $w = (c_0, \dots, c_n, 0_C, 0_C, 0_C, \dots) \in C^{\S}$ such that $f(c_j) = a_j$ for all $j \in \{0, \dots, n\}$. Then

$$\begin{aligned}
h(x) &= \Big((gHG^k \mathrm{int}_{[F,G]})(v)\Big)_{k\in\mathbb{N}} \\
&= \left(gHG^k \left(\sum_{j=0}^{n} (G^j F f)(c_j) \right) \right)_{k\in\mathbb{N}} \\
&= \left(gH \left(\sum_{j=0}^{n} (G^{k+j} F f)(c_j) \right) \right)_{k\in\mathbb{N}} \\
&= \Big(\Big(gH \mathrm{int}_{[F,G]} f^{\S} \mathsf{sh}_C^k\Big)(w) \Big)_{k\in\mathbb{N}} \\
&= ((\psi\, \mathsf{sh}_C^k)(w))_{k\in\mathbb{N}} \\
&= b_\psi(w).
\end{aligned}$$

So the map h is in fact a map $h\colon X \to \mathrm{Im}(b_\psi)$. Now, we claim that the diagram

$$\begin{tikzcd}
A \arrow[r, "F"] & X \arrow[r, "G"] \arrow[d, "h"] & X \arrow[r, "H"] \arrow[d, "h"] & B \arrow[d, "g"] \\
C \arrow[u, two heads, "f"] \arrow[r, "e_\psi \circ \mathsf{in}_C"'] & \mathrm{Im}(b_\psi) \arrow[r, "\mathsf{tl}'_\psi"'] & \mathrm{Im}(b_\psi) \arrow[r, "\mathsf{hd}_D \circ m_\psi"'] & D
\end{tikzcd}$$

in **Vect** commutes. Indeed, for $c \in C$,

$$\begin{aligned}
(e_\psi\, \mathsf{in}_C)(c) &= b_\psi(c, 0_C, 0_C, 0_C, \dots) \\
&= \Big((\psi\, \mathsf{sh}_C^k)(c, 0_C, 0_C, 0_C, \dots)\Big)_{k\in\mathbb{N}} \\
&= \Big((gH \mathrm{int}_{[F,G]} f^{\S} \mathsf{sh}_C^k)(c, 0_C, 0_C, 0_C, \dots)\Big)_{k\in\mathbb{N}} \\
&= \Big((gH \mathrm{int}_{[F,G]} \mathsf{sh}_A^k)(f(c), 0_C, 0_C, 0_C, \dots)\Big)_{k\in\mathbb{N}} \\
&= \Big((gHG^k F f)(c)\Big)_{k\in\mathbb{N}} \\
&= (hFf)(c).
\end{aligned}$$

So the square on the left commutes. For the middle square, we see that for any $x \in X$,

$$\begin{aligned}
(hG)(x) &= \Big((gHG^k)(Gx)\Big)_{k\in\mathbb{N}} \\
&= \Big((gHG^{k+1})(x)\Big)_{k\in\mathbb{N}}
\end{aligned}$$

$$
\begin{aligned}
&= \mathsf{tl}'_\psi\Big((gHG^k)(x)\Big)_{k\in\mathbb{N}} \\
&= (\mathsf{tl}'_\psi h)(x).
\end{aligned}
$$

For the square on the right, we have

$$
\begin{aligned}
(\mathsf{hd}_D m_\psi h)(x) &= \mathsf{hd}_D\Big((gHG^k)(x)\Big)_{k\in\mathbb{N}} \\
&= (gH)(x)
\end{aligned}
$$

for all $x \in X$. So (f, g, h) is a morphism from $(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B)$ to $\mathcal{K}(C^\S \xrightarrow{\psi} D)$ in **RLDS**. Moreover, such h is unique as it is uniquely determined by the middle and right-hand squares above.

Conversely, suppose we have a morphism $(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B)$ to $\mathcal{K}(C^\S \xrightarrow{\psi} D)$ in **RLDS**. So the diagram

$$
\begin{tikzcd}
A \arrow[r, "F"] & X \arrow[r, "G"] \arrow[d, "h"] & X \arrow[r, "H"] \arrow[d, "h"] & B \arrow[d, "g"] \\
C \arrow[u, two heads, "f"] \arrow[r, "e_\psi \circ \mathsf{in}_C"'] & \mathrm{Im}(b_\psi) \arrow[r, "\mathsf{tl}'_\psi"'] & \mathrm{Im}(b_\psi) \arrow[r, "\mathsf{hd}_D \circ m_\psi"'] & D
\end{tikzcd}
$$

in **Vect** commutes. Then for any $c \in C$ and $k \in \mathbb{N}$, we have that

$$
\begin{aligned}
(gHG^kFf)(c) &= (\mathsf{hd}_D m_\psi h G^k F f)(c) \\
&= (\mathsf{hd}_D m_\psi (\mathsf{tl}'_\psi)^k hFf)(c) \\
&= (\mathsf{hd}_D m_\psi (\mathsf{tl}'_\psi)^k e_\psi \mathsf{in}_C)(c) \\
&= (\psi\, \mathsf{sh}_C^k\, \mathsf{in}_C)(c).
\end{aligned}
$$

Thus, for any $n \in \mathbb{N}$ and $c_0, \dots, c_n \in C$, we have

$$
\begin{aligned}
\psi(c_0, \dots, c_n, 0_C, 0_C, 0_C, \dots) &= \sum_{k=0}^{n} (\psi\, \mathsf{sh}_C^k\, \mathsf{in}_C)(c_k) \\
&= \sum_{k=0}^{n} (gHG^kFf)(c_k) \\
&= gH\left(\sum_{k=0}^{n} (G^kFf)(c_k)\right) \\
&= (gH\mathrm{int}_{[F,G]} f^\S)(c_0, \dots, c_n, 0_C, 0_C, 0_C, \dots).
\end{aligned}
$$

Therefore the diagram

$$\begin{tikzcd}
A + A^{\S} \arrow[r, "\mathrm{id}_A + \mathrm{int}_{[F,G]}"] \arrow[d, "{[\mathsf{in}_A, \mathsf{sh}_A]}"', "\cong"] & A + X \arrow[d, "{[F,G]}"] & \\
A^{\S} \arrow[r, "\mathrm{int}_{[F,G]}"'] & X \arrow[r, "H"'] & B \arrow[d, "g"] \\
C^{\S} \arrow[u, "f^{\S}"] \arrow[rr, "\psi"'] & & D
\end{tikzcd}$$

in **Vect** commutes, making (f,g) a morphism from $\mathcal{B}(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B)$ to $(C^{\S} \xrightarrow{\psi} D)$ in **RLB**.

Therefore the assignment which maps a morphism $(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B) \xrightarrow{(f,g,h)} \mathcal{K}(C^{\S} \xrightarrow{\psi} D)$ in **RLDS** to the morphism $\mathcal{B}(A \xrightarrow{F} X \xrightarrow{G} X \xrightarrow{H} B) \xrightarrow{(f,g)} (C^{\S} \xrightarrow{\psi} D)$ is a bijection. Furthermore this bijection is natural in both the domain and codomain of the morphisms, simply because compositions in **RLDS** and **RLB** are defined by compositions of the components of the morphisms. So we indeed have the adjunction $\mathcal{B} \dashv \mathcal{K}$.

□

Exercise 2.5.17

This exercise describes the so-called terms-as-natural-transformations view which originally stems from Lawvere (1963). It is elaborated in a coalgebraic context in Kurz and Rosický (2005).

*Let $H\colon$ **Sets** $\to$ **Sets** be a (not necessarily polynomial) functor with free algebras, given by a left adjoint F to the forgetful functor $U\colon$ **Alg**$(H) \to$ **Sets**. Let X be an arbitary set whose elements are considered as variables. Elements of the carrier $UF(X)$ of the free algebra on X can then be seen as terms containing free variables from X. Show that there is a bijective correspondence:*

$$\frac{\text{terms } t \in UF(X)}{\overline{\text{natural transformations } \tau\colon U^X \Rightarrow U}}\,.$$

Hint: The component of the natural transformation at a specific algebra $HA \to A$ is the mapping which takes a valuation $\rho\colon X \to A$ of the variables in A to an interpretation $[\![t]\!]^A_\rho$ of the term t in the algebra A. Naturality then says that for a homomorphism $f\colon A \to B$ of algebras, one has the familiar equation $f([\![t]\!]^A_\rho) = [\![t]\!]^B_{f\circ\rho}$.

Solution. Fix a set X and an H-algebra $\alpha\colon HA \to A$. Let $\varphi_X\colon HFX \to FX$ denote the free H-algebra on X. From the adjunction $\begin{tikzcd} \mathbf{Sets} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Alg}(H) \arrow[l, bend left, "U"] \end{tikzcd}$, we get a bijection $\theta_{X,(A,\alpha)}\colon \hom_{\mathbf{Sets}}(X,A) \xrightarrow{\cong} \hom_{\mathbf{Alg}(H)}((FX,\varphi_X),(A,\alpha))$ which is natural in X and (A,α). Fix a term $t \in U(FX,\varphi_X)$, and define

$$\tau^t_{(A,\alpha)}(\rho) := \theta_{X,(A,\alpha)}(\rho)(t)$$

for all functions $\rho\colon X \to A$, so that $\tau^t_{(A,\alpha)}$ is a function from A^X to A. We claim that τ^t is a natural transformation from U^X to U. Indeed, suppose we have an H-algebra homomorphism $(A,\alpha) \xrightarrow{f} (B,\beta)$.

Then the square

$$\begin{tikzcd}
\hom_{\mathbf{Sets}}(X, A) \arrow[r, "\theta_{X,(A,\alpha)}", "\cong"'] \arrow[d, "f \circ (-)"'] & \hom_{\mathbf{Alg}(H)}((FX, \varphi_X), (A, \alpha)) \arrow[d, "f \circ (-)"] \\
\hom_{\mathbf{Sets}}(X, B) \arrow[r, "\cong", "\theta_{X,(B,\beta)}"'] & \hom_{\mathbf{Alg}(H)}((FX, \varphi_X), (B, \beta))
\end{tikzcd}$$

in **Sets** commutes, by the naturality of the adjunction bijection θ. We want to show that the square

$$\begin{tikzcd}
A^X \arrow[r, "f^X"] \arrow[d, "\tau^t_{(A,\alpha)}"'] & B^X \arrow[d, "\tau^t_{(B,\beta)}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

in **Sets** commutes. Indeed, for any function $\rho\colon X \to A$, we have

$$\begin{aligned}
\left(f \circ \tau^t_{(A,\alpha)}\right)(\rho) &= f\Big(\theta_{X,(A,\alpha)}(\rho)(t)\Big) \\
&= \theta_{X,(B,\beta)}\Big((f\rho)(t)\Big) \\
&= \tau^t_{(B,\beta)}(f\rho) \\
&= \left(\tau^t_{(B,\beta)} \circ f^X\right)(\rho).
\end{aligned}$$

Now we show that the assignment $UFX \ni t \mapsto \tau^t$ is a bijection between the set UFX and the collection of natural transformations from U^X to U. Let $\eta\colon \mathrm{id}_{\mathbf{Sets}} \Rightarrow UF$ be the unit of the adjunction $F \dashv U$. Then for any H-algebra (A, α) and for any function $\rho\colon X \to A$, the morphism $FX \xrightarrow{\theta_{X,(A,\alpha)}(\rho)} A$ is the unique H-algebra homomorphism making the diagram

$$\begin{tikzcd}
HFX \arrow[rr, "H\left(\theta_{X,(A,\alpha)}(\rho)\right)"] \arrow[d, "\varphi_X"'] & & HA \arrow[d, "\alpha"] \\
FX \arrow[rr, "\theta_{X,(A,\alpha)}(\rho)"] & & A \\
& X \arrow[ul, "\eta_X"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

in **Sets** commute. Then, for any natural transformation $\nu\colon U^X \Rightarrow U$, for any H-algebra (A, α), and for any function $\rho\colon X \to A$, the commutativity of the square

$$\begin{tikzcd}
(FX)^X \arrow[r, "\left(\theta_{X,(A,\alpha)}(\rho)\right)^X"] \arrow[d, "\nu_{FX}"'] & A^X \arrow[d, "\nu_A"] \\
FX \arrow[r, "\theta_{X,(A,\alpha)}(\rho)"'] & A
\end{tikzcd}$$

in **Sets** asserts that

$$\begin{aligned}
\nu_A(\rho) &= \nu_A\big(\theta_{X,(A,\alpha)}(\rho) \circ \eta_X\big) \\
&= \nu_A\Big(\big(\theta_{X,(A,\alpha)}(\rho)\big)^X(\eta_X)\Big) \\
&= \theta_{X,(A,\alpha)}(\rho)\big(\nu_{FX}(\eta_X)\big) \\
&= \tau^{\nu_{FX}(\eta_X)}_{(A,\alpha)}(\rho).
\end{aligned}$$

Thus ν is completely and uniquely determined by the value of $\nu_{FX}(\eta_X) \in UFX$. From this, we can conclude that the assignment $UFX \ni t \mapsto \tau^t$ is indeed a bijection from UFX to the collection of natural transformations $U^X \Rightarrow U$. $\square$

3 Bisimulations

3.1 Relation Lifting, Bisimulations and Congruences

Exercise 3.1.1

Use the description (2.17) of a list functor $F^\star$ to show that:

$$\mathrm{Rel}(F^\star)(R) = \{\, (\langle u_1, \ldots, u_n\rangle, \langle v_1, \ldots, v_n\rangle) \mid \forall i \leq n. \mathrm{Rel}(F)(R)(u_i, v_i) \,\}.$$

Solution. We recall, from Equation (2.17), that $F^\star = \coprod_{n\in\mathbb{N}} (F(-))^n = \coprod_{n\in\mathbb{N}} (\underbrace{F \times \cdots \times F}_{n \text{ copies of } F})$. So, for any sets X and Y and any relation $R \subseteq X \times Y$, we have

$$\begin{aligned}
\mathrm{Rel}(F^\star)(R) &= \mathrm{Rel}\left(\coprod_{n\in\mathbb{N}} (F(-))^n\right)(R) \\
&= \bigcup_{n\in\mathbb{N}} \{(x,y) \ : \ x \in (FX)^n,\ y \in (FY)^n, \text{ and } \mathrm{Rel}((F(-))^n)(x,y)\} \\
&= \bigcup_{n\in\mathbb{N}} \Big\{(\langle u_1,\ldots,u_n\rangle, \langle v_1,\ldots,v_n\rangle) \ : \ \langle u_1,\ldots,u_n\rangle \in (FX)^n,\ \langle v_1,\ldots,v_n\rangle \in (FY)^n, \\
&\qquad\qquad \text{and } \mathrm{Rel}((F(-))^n)(\langle u_1,\ldots,u_n\rangle, \langle v_1,\ldots,v_n\rangle)\Big\} \\
&= \bigcup_{n\in\mathbb{N}} \Big\{(\langle u_1,\ldots,u_n\rangle, \langle v_1,\ldots,v_n\rangle) \ : \ \langle u_1,\ldots,u_n\rangle \in (FX)^n,\ \langle v_1,\ldots,v_n\rangle \in (FY)^n, \text{ and} \\
&\qquad\qquad \text{for all } i \in \{1,\ldots,n\rangle, \text{ we have } \mathrm{Rel}(F)(R)(u_i,v_i)\Big\},
\end{aligned}$$

where we have used the product and coproduct clauses in the inductive definition of relation lifting. □

Exercise 3.1.2

Unfold the definition for bisimulation for various kind of tree coalgebras, like $X \to 1 + (A \times X \times X)$ and $X \to (A \times X)^\star$.

Solution. Fix a set A. Let $F\colon \mathbf{Sets} \to \mathbf{Sets}$ be the functor defined on objects by $FX \coloneqq 1 + (A \times X \times X)$ for all $X \in \mathbf{Sets}$. Fix two F-coalgebras (X, c) and (Y, d). Then for any relation $R \subseteq X \times Y$, any $x \in X$, and any $y \in Y$, we have that $(c(x), d(y)) \in \mathrm{Rel}(F)(R)$ if and only if either of the following hold:

- $c(x) = d(y) \in 1$, or
- $c(x) \in A \times X \times X$, $d(y) \in A \times Y \times Y$, and writing $c(x) = (a, x_1, x_2)$ and $d(y) = (b, y_1, y_2)$, we have $a = b$ and both $R(x_1, y_1)$ and $R(x_2, y_2)$.

So a relation $R \subseteq X \times Y$ is a bisimulation for c and d if and only if $R(x, y)$ implies that either of the above hold for each $x \in X$ and $y \in Y$.

Continue fixing a set A, and let $G\colon \mathbf{Sets} \to \mathbf{Sets}$ be the functor defined on objects by $GX \coloneqq (A \times X)^\star = \coprod_{n\in\mathbb{N}} (A \times X)^n$ for all $X \in \mathbf{Sets}$. Fix two G-coalgebras (X, c) and (Y, d). Then for any relation $R \subseteq X \times Y$, any $x \in X$, and any $y \in Y$, we have that $(c(x), d(y)) \in \mathrm{Rel}(G)(R)$ if and only if when we write $c(x) = ((a_1, x_1), \ldots, (a_n, x_n))$ and $d(y) = ((b_1, y_1), \ldots, (b_k, y_k))$, we have that $n = k$ and for each $i \in \{1, \ldots, n\}$, both $a_i = b_i$ and $R(x_i, y_i)$ hold. Thus a relation $R \subseteq X \times Y$ is a bisimulation for c and d if and only if $R(x, y)$ implies said properties for each $x \in X$ and $y \in Y$. □

Exercise 3.1.3
Do the same for classes in object-oriented languages (see (1.10)), described as coalgebras of a functor in Exercise 2.3.6.3.

Solution. #?? ☐

Exercise 3.1.4
#??

Solution. #?? ☐

Exercise 3.1.5
#??

Solution. #?? ☐

Exercise 3.1.6
#??

Solution. #?? ☐

3.2 Properties of Bisimulations

Exercise 3.2.1
#??

Solution. #?? □

Exercise 3.2.2
#??

Solution. #?? □

Exercise 3.2.3
#??

Solution. #?? □

Exercise 3.2.4
#??

Solution. #?? □

Exercise 3.2.5
#??

Solution. #?? □

Exercise 3.2.6
#??

Solution. #?? □

Exercise 3.2.7
#??

Solution. #?? □

3.3 Bisimulations as Spans and Cospans

Exercise 3.3.1
#??

Solution. #?? □

Exercise 3.3.2
#??

Solution. #?? □

Exercise 3.3.3
#??

Solution. #?? □

Exercise 3.3.4
#??

Solution. #?? □

3.4 Bisimulations and the Coinduction Proof Principle

Exercise 3.4.1
#??

Solution. #?? ☐

Exercise 3.4.2
#??

Solution. #?? ☐

Exercise 3.4.3
#??

Solution. #?? ☐

Exercise 3.4.4
#??

Solution. #?? ☐

Exercise 3.4.5
#??

Solution. #?? ☐

Exercise 3.4.6
#??

Solution. #?? ☐

Exercise 3.4.7
#??

Solution. #?? ☐

3.5 Process Semantics

Exercise 3.5.1
#??

Solution. #?? $\square$

Exercise 3.5.2
#??

Solution. #?? $\square$

Exercise 3.5.3
#??

Solution. #?? $\square$

Exercise 3.5.4
#??

Solution. #?? $\square$

4 Logic, Lifting and Finality

4.1 Multiset and Distribution Functors

Exercise 4.1.1
#??

Solution. #?? □

Exercise 4.1.2
#??

Solution. #?? □

Exercise 4.1.3
#??

Solution. #?? □

Exercise 4.1.4
#??

Solution. #?? □

Exercise 4.1.5
#??

Solution. #?? □

Exercise 4.1.6
#??

Solution. #?? □

Exercise 4.1.7
#??

Solution. #?? □

Exercise 4.1.8
#??

Solution. #?? □

4.2 Weak Pullbacks

Exercise 4.2.1

#??

Solution. #?? □

Exercise 4.2.2

Consider in an arbitrary category a pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Prove that if m is a mono, then so is m'.

Solution. Fix a category $\mathbb{C}$. Suppose we have a pullback square

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "m"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

in $\mathbb{C}$, with m monic. For a parallel pair of morphisms $D \underset{k}{\overset{h}{\rightrightarrows}} P$ with $m'h = m'k$, we have

$$mfh = gm'h = gm'k = mfk.$$

So $fh = fk$, since m is a monomorphism. The uniqueness clause in the universal property of pullbacks now lets us conclude that $h = k$. □

Exercise 4.2.3

#??

Solution. #?? □

Exercise 4.2.4

#??

Solution. #?? □

Exercise 4.2.5

#??

Solution. #?? □

Exercise 4.2.6

The following two results are known as the pullback lemmas. Prove them yourself.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(B)}"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(A)}"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

1. *If* (A) *and* (B) *are pullback squares, then the outer rectangle is also a pullback square.*

2. *If the outer rectangle and* (A) *are pullback squares, then* (B) *is a pullback square as well.*

Solution. Fix a category $\mathbb{C}$.

(i) Suppose we have two pullback squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "\ell"] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "m"] \\ C \arrow[r, "k"'] & D \arrow[r, "n"'] & F \end{tikzcd}$$

in $\mathbb{C}$. Suppose we have an object $P \in \mathbb{C}$ and two morphisms $C \xleftarrow{q} P \xrightarrow{p} E$ in $\mathbb{C}$ such that $mp = nkq$. Then the morphisms $C \xleftarrow{kq} P \xrightarrow{p} E$ yield a unique morphism $P \xrightarrow{r} B$ such that $gr = kq$ and $\ell r = p$. Then the morphisms $C \xleftarrow{q} P \xrightarrow{r} B$ yield a unique morphism $P \xrightarrow{s} A$ such that $hs = q$ and $fs = r$. We then see that $\ell f s = \ell r = p$.

$$\begin{tikzcd} P \arrow[dr, "s"] \arrow[drr, bend left, "r"] \arrow[drrr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & & \\ & A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "\ell"] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "m"] \\ & C \arrow[r, "k"'] & D \arrow[r, "n"'] & F \end{tikzcd}$$

Suppose we are given another morphism $P \xrightarrow{t} A$ satisfying $ht = q$ and $\ell f t = p$. Then $gft = kht = kq$. Together with the equality $\ell f t = p$, it follows that $ft = r$. This, together with the equality $ht = q$, allows us to conclude that $t = s$.

(ii) Suppose that we have a commuting diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "\ell"] \arrow[d, "g"] & E \arrow[d, "m"] \\ C \arrow[r, "k"'] & D \arrow[r, "n"'] & F \end{tikzcd}$$

in $\mathbb{C}$ such that the following two squares

$$\begin{tikzcd} A \arrow[r, "\ell f"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "m"] \\ C \arrow[r, "nk"'] & F \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\ell"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "m"] \\ D \arrow[r, "n"'] & F \end{tikzcd}$$

are pullback squares in $\mathbb{C}$. Suppose we have an object $P \in \mathbb{C}$ and two morphisms $A \xleftarrow{q} P \xrightarrow{p} B$ in $\mathbb{C}$ such that $gp = kq$. Then the two morphisms $D \xleftarrow{kq} P \xrightarrow{\ell p} B$ satisfy $m(\ell p) = n(kq)$. Using the fact that the outer rectangle is a pullback, there exists a unique morphism $P \xrightarrow{r} A$ such that $\ell f r = \ell p$ and $hr = q$. We then see that the parallel pair morphisms $P \underset{fr}{\overset{p}{\rightrightarrows}} B$ satisfy $\ell p = \ell f r$ and $gp = kq = khr = gfr$. The fact that the right-hand square is a pullback now implies that $p = fr$.

$$
\begin{tikzcd}
P \arrow[dr, "r"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & & \\
& A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "\ell"] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "m"] \\
& C \arrow[r, "k"'] & D \arrow[r, "n"'] & F
\end{tikzcd}
$$

Given any other morphism $P \xrightarrow{s} A$ satisfying $fs = p$ and $hs = q$, we also obtain the equality $\ell f s = \ell p$. Therefore $s = r$. $\square$

Exercise 4.2.7

Let $F \colon \mathbb{C} \to \mathbb{C}$ be an endofunctor on a cateogry $\mathbb{C}$ with pullbacks. Prove that the category $\mathbf{Alg}(F)$ of algebras also has pullbacks, constructed as in $\mathbb{C}$.

Solution. Suppose we have a cospan

$$
\begin{tikzcd}
& (A, \alpha) \arrow[d, "f"] \\
(B, \beta) \arrow[r, "g"'] & (C, \gamma)
\end{tikzcd}
$$

in $\mathbf{Alg}(F)$. We take the pullback

$$
\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}
$$

in $\mathbb{C}$, yielding the commuting diagram

$$
\begin{tikzcd}
FP \arrow[rrr, "Fp_1"] \arrow[ddd, "Fp_2"'] \arrow[dr, dashed, "\exists ! \pi"] & & & FA \arrow[dl, "\alpha"'] \arrow[ddd, "Ff"] \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] & \\
& B \arrow[r, "g"'] & C & \\
FB \arrow[ur, "\beta"] \arrow[rrr, "Fg"'] & & & FC \arrow[ul, "\gamma"']
\end{tikzcd}
$$

in $\mathbb{C}$.

We claim that the F-algebra (P, π) is the (object of the) desired pullback. For any two F-algebra homomorphisms $(B, \beta) \xleftarrow{k} (X, \xi) \xrightarrow{h} (A, \alpha)$ satisfying $fh = gk$, we have a unique morphism $X \xrightarrow{t} P$ in $\mathbb{C}$ satisfying $p_1 t = h$ and $p_2 t = k$. We now need to check that this t really is a homomorphism of F-algebras from (X, ξ) to (P, π). Indeed,

$$p_1 t\xi = h\xi = \alpha Fh = \alpha(Fp_1)(Ft) = p_1 \pi Ft$$

and, similarly, $p_2 t\xi = p_2 \pi Ft$. So $t\xi = \pi Ft$. □

Exercise 4.2.8
#??

Solution. #?? □

Exercise 4.2.9
#??

Solution. #?? □

Exercise 4.2.10
#??

Solution. #?? □

Exercise 4.2.11
#??

Solution. #?? □

4.3 Predicates and Relations

Exercise 4.3.1

Let $(\mathfrak{M}, \mathfrak{E})$ be a logical factorisation system.

1. *Show that a map $f \in \mathfrak{M} \cap \mathfrak{E}$ is an isomorphism.*
2. *Prove that if we can factor a map g both as $g = m \circ e$ and as $g = m' \circ e'$, where $m, m' \in \mathfrak{M}$ and $e, e' \in \mathfrak{E}$, then there is a unique isomorphism φ with $m' \circ \varphi = m$ and $\varphi \circ e = e'$.*
3. *Show for $m \in \mathfrak{M}$ and $e \in \mathfrak{E}$ that $\mathfrak{m}(m \circ f) = m \circ \mathfrak{m}(f)$ and $\mathfrak{e}(f \circ e) = \mathfrak{m}(f) \circ e$, where $\mathfrak{m}(-)$ and $\mathfrak{e}(-)$ take the $\mathfrak{M}$-part and the $\mathfrak{E}$-part as in Definition 4.3.2.2.*

Solution. Let $\mathbb{C}$ be a category with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$.

1. Let $(A \xrightarrow{f} B) \in \mathfrak{M} \cap \mathfrak{E}$. We have the commuting diagram

$$\begin{tikzcd}
A \arrow[r, tail, two heads, "f"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[d, "\mathrm{id}_B"] \arrow[dl, dashed, "\exists! g"'] \\
A \arrow[r, tail, two heads, "f"'] & B
\end{tikzcd}$$

   in $\mathbb{C}$, by the diagonal-fill-in property. Thus f is an isomorphism with inverse g.

2. Suppose we have two factorisations $g = m \circ e = m' \circ e'$ of a morphism $A \xrightarrow{g} B$ in $\mathbb{C}$, where $(I \xrightarrow{m} B), (I' \xrightarrow{m'} B) \in \mathfrak{M}$ and $(A \xrightarrow{e} I), (A \xrightarrow{e'} I') \in \mathfrak{E}$. That is, the diagram

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, two heads, "e'"'] \arrow[dr, "g"] & I \arrow[d, tail, "m"] \\
I' \arrow[r, tail, "m'"'] & B
\end{tikzcd}$$

   in $\mathbb{C}$ commutes. Then, by the diagonal-fill-in property, there exists a unique morphism $I \xrightarrow{\varphi} I'$ in $\mathbb{C}$ such that $\varphi e = e'$ and $m'\varphi = m$. Similarly, there exists a unique morphism $I' \xrightarrow{\varphi'} I$ such that $\varphi' e' = e$ and $m\varphi' = m'$. So

$$m\varphi'\varphi = (m\varphi')\varphi = m'\varphi = m$$

   yielding $\varphi'\varphi = \mathrm{id}_I$ since m is monic. Similarly, $\varphi\varphi' = \mathrm{id}_{I'}$. So $I \underset{\cong}{\xrightarrow{\varphi}} I'$ is the desired unique isomorphism.

3. Fix an arbitrary morphism $B \xrightarrow{f} C$, an abstract monomorphism $C \xrightarrow{m} D$, and an abstract epimorphism $A \xrightarrow{e} B$. We see that $mf = \mathfrak{e}(mf) \circ \mathfrak{m}(mf) = \mathfrak{e}(f) \circ \mathfrak{m}(f) \circ m$. By the previous part, we have $\mathfrak{m}(mf) = \mathfrak{m}(f) \circ m$, where we actually mean that these two morphisms live in the same equivalence class of subobjects of D. Allowing ourselves to assume that the relevant

subobjects coincide, i.e. $m \circ \mathfrak{m}(f) = \mathfrak{m}(mf)$ and $\mathfrak{m}(f) = \mathfrak{m}(fe)$, we get the commuting diagram

$$\begin{tikzcd}
A \arrow[rr, "\mathfrak{e}(fe)"] \arrow[dr, "e"] & & I \arrow[dd, "\mathfrak{m}(fe)"] & \\
& B \arrow[rr, "\mathfrak{e}(mf)"'] \arrow[d, "\mathfrak{e}(f)"'] \arrow[dr, "f"] & & I \arrow[dd, "\mathfrak{m}(mf)"] \\
& I \arrow[r, "\mathfrak{m}(f)"'] & C \arrow[dr, "m"] & \\
& & & D
\end{tikzcd}$$

in $\mathbb{C}$. As the appropriate diagonal-fill-ins must be the id_I, we obtain $\mathfrak{e}(f) \circ e = \mathfrak{e}(fe)$. □

Exercise 4.3.2

Let $F \colon \mathbb{C} \to \mathbb{C}$ be an endofunctor on a category $\mathbb{C}$ with a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$.

1. *Assume that F preserves abstract epis, i.e. $e \in \mathfrak{E} \implies F(e) \in \mathfrak{E}$. Prove that the category $\mathbf{Alg}(F)$ of algebras also carries a logical factorisation system. Use that pullbacks in $\mathbf{Alg}(F)$ are constructed as in $\mathbb{C}$; see Exercise 4.2.7.*

2. *Check that every endofunctor $F \colon \mathbf{Sets} \to \mathbf{Sets}$ satisfies this assumption, i.e. preserves surjections — if the axiom of choice holds. Hint: Recall that the axiom of choice can be formulated as: each surjection has a section; see Section 2.1.*

Solution.

1. Let $\mathfrak{M}'$ and $\mathfrak{E}'$ be the collections of morphisms in $\mathbf{Alg}(F)$ for which the underlying morphism is in $\mathfrak{M}$ and $\mathfrak{E}$ respectively. It is clear that both $\mathfrak{M}'$ and $\mathfrak{E}'$ contain all isomorphisms from $\mathbb{C}$ and are both closed under composition.

   Suppose we have an F-algebra homomorphism $(FA \xrightarrow{\alpha} A) \xrightarrow{f} (FB \xrightarrow{\beta} B)$. This yields the commuting diagram

$$\begin{tikzcd}
& F(\mathrm{Im}(f)) \arrow[dr, "F(\mathfrak{m}(f))"] & \\
FA \arrow[ur, "F(\mathfrak{e}(f))\in\mathfrak{E}"] \arrow[rr, "Ff"'] \arrow[d, "\alpha"'] & & FB \arrow[d, "\beta"] \\
A \arrow[rr, "f"] \arrow[dr, "\mathfrak{e}(f)"'] & & B \\
& \mathrm{Im}(f) \arrow[ur, "\mathfrak{m}(f)"'] &
\end{tikzcd}$$

   in $\mathbb{C}$. So there exists a unique morphism $F(\mathrm{Im}(f)) \xrightarrow{\gamma} \mathrm{Im}(f)$ making the diagram

$$\begin{tikzcd}
FA \arrow[r, "F(\mathfrak{e}(f))\in\mathfrak{E}"] \arrow[d, "\alpha"'] & F(\mathrm{Im}(f)) \arrow[d, "F(\mathfrak{m}(f))"] \arrow[ddl, "\gamma"'] \\
A \arrow[d, "\mathfrak{e}(f)"'] & FB \arrow[d, "\beta"] \\
\mathrm{Im}(f) \arrow[r, "\mathfrak{m}(f)"'] & B
\end{tikzcd}$$

in $\mathbb{C}$ commute, by the diagonal-fill-in property. So get the commuting triangle

$$\begin{tikzcd}
(A,\alpha) \arrow[r, "\mathfrak{e}(f)"] \arrow[dr, "f"'] & (\mathrm{Im}(f),\gamma) \arrow[d, "\mathfrak{m}(f)"] \\
& (B,\beta)
\end{tikzcd}$$

in $\mathbf{Alg}(F)$. So we have factored every morphism in $\mathbf{Alg}(F)$ into a composition of a morphism in $\mathfrak{E}'$ followed by a morphism in $\mathfrak{M}'$.

A commutative square
$$\begin{tikzcd}
(A,\alpha) \arrow[r, "e\in\mathfrak{E}'"] \arrow[d, "f"'] & (B,\beta) \arrow[d, "g"] \\
(C,\gamma) \arrow[r, "m\in\mathfrak{M}'"'] & (D,\delta)
\end{tikzcd}$$
in $\mathbf{Alg}(F)$ yields the commutative square

$$\begin{tikzcd}
A \arrow[r, "e\in\mathfrak{E}"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "\exists! t"'] \\
C \arrow[r, "m\in\mathfrak{M}"'] & D
\end{tikzcd}$$
in $\mathbb{C}$, by the diagonal-fill-in property of $(\mathfrak{M}, \mathfrak{E})$. This t is an F-algebra homomorphism from (B,β) to (C,γ) because of the string of equalities

$$mt\beta = g\beta = \delta(Fg) = \delta(Fm)(Ft) = m\gamma(Ft).$$

in $\mathbb{C}$, from which $t\beta = \gamma(Ft)$ follows due to m being monic.

A morphism $X \xrightarrow{f} Y$ in an arbitrary category $\mathbb{D}$ is a monomorphism if and only if the (commutative) square

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

in $\mathbb{D}$ is a pullback square. As pullbacks in $\mathbf{Alg}(F)$ are formed at the level of $\mathbb{C}$, monomorphisms in $\mathbf{Alg}(F)$ are precisely the F-algebra homomorphisms whose underlying morphisms are monomorphisms in $\mathbb{C}$. Therefore all morphisms in $\mathfrak{M}'$ are monic.

Finally, that $\mathfrak{M}'$ is closed under pullback, and that $\mathfrak{E}'$ is closed under pullbacks of morphisms in $\mathfrak{M}'$, follows simply from pullbacks in $\mathbf{Alg}(F)$ being constructed on the level of $\mathbb{C}$.

2. Fix an endofunctor $F\colon \mathbf{Sets} \to \mathbf{Sets}$ and let $e\colon X \to Y$ be a surjective function. Choose a function $s\colon Y \to X$ satisfying $es = \mathrm{id}_Y$, i.e. s is a section of e. Then $(Fe)(Fs) = F(es) = F\mathrm{id}_Y = \mathrm{id}_{FY}$. Therefore $Fe\colon FX \to FY$ is split epic in $\mathbf{Sets}$, and is hence a surjection. $\square$

Exercise 4.3.3

Define the category $\mathrm{EnRel}(\mathbb{C})$ *of endorelations in a category* $\mathbb{C}$ *(with a logical factorisation system) via* *the following pullback of functors:*

$$
\begin{tikzcd}
\mathrm{EnRel}(\mathbb{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Rel}(\mathbb{C}) \arrow[d] \\
\mathbb{C} \arrow[r, "\langle \mathrm{id}_{\mathbb{C}}, \mathrm{id}_{\mathbb{C}} \rangle"'] & \mathbb{C} \times \mathbb{C}
\end{tikzcd}
$$

1. *Describe this category* $\mathrm{EnRel}(\mathbb{C})$ *in detail.*

2. *Demonstrate that taking equality relations on an object forms a functor* $\mathrm{Eq}(-)\colon \mathbb{C} \to \mathrm{EnRel}(\mathbb{C})$.

Solution.

1. The objects of $\mathrm{EnRel}(\mathbb{C})$ are relations $R \overset{r}{\rightarrowtail} X \times X$ whose codomain is the product of an object in $\mathbb{C}$ with itself. A morphism in $\mathrm{EnRel}(\mathbb{C})$ from $R \overset{r}{\rightarrowtail} X \times X$ to $S \overset{r}{\rightarrowtail} Y \times Y$ is a morphism $f\colon X \to Y$ such that the diagram

$$
\begin{tikzcd}
R \arrow[r, dashed, "\exists!"] \arrow[d, tail, "r"'] & S \arrow[d, tail, "s"] \\
X \times X \arrow[r, "f \times f"'] & Y \times Y
\end{tikzcd}
$$

   in $\mathbb{C}$ commutes. Composition and identities in $\mathrm{EnRel}(\mathbb{C})$ are exactly as in $\mathbb{C}$.

2. The operation $\mathrm{Eq}\colon \mathbb{C} \to \mathrm{EnRel}(\mathbb{C})$ is defined on morphisms simply as $\mathrm{Eq} f \coloneqq f$ for all morphisms f in $\mathbb{C}$. To see that this indeed sends a morphism in $\mathbb{C}$ to a morphism in $\mathrm{EnRel}(\mathbb{C})$, we use the diagonal-fill-in property: for a morphism $X \xrightarrow{f} Y$ in $\mathbb{C}$, we have the commuting diagram

$$
\begin{tikzcd}
X \arrow[r, "\mathfrak{e}(\langle \mathrm{id}_X, \mathrm{id}_X \rangle)"] \arrow[d, "f"'] & \mathrm{Eq}X \arrow[d, "\mathfrak{m}(\langle \mathrm{id}_X, \mathrm{id}_X \rangle)"] \arrow[ddl, dashed, "\exists!"'] \\
Y \arrow[d, "\mathfrak{e}(\langle \mathrm{id}_Y, \mathrm{id}_Y \rangle)"'] & X \times X \arrow[d, "f \times f"] \\
\mathrm{Eq}Y \arrow[r, "\mathfrak{m}(\langle \mathrm{id}_Y, \mathrm{id}_Y \rangle)"'] & Y \times Y
\end{tikzcd}
$$

   in $\mathbb{C}$. The functoriality of $\mathrm{Eq}\colon \mathbb{C} \to \mathrm{EnRel}(\mathbb{C})$ is clear from definition of Eq. □

Exercise 4.3.4
Let $\mathbb{C}$ *be a category with a logical factorisation system and finite coproducts* $(0, +)$.

1. *Show that the image of the unique map* $!\colon 0 \to X$ *is the least element* $\perp$ *in the poset* $\mathrm{Pred}(X)$ *of predicates on* X.

2. *Similarly, show that the join* $m \vee n$ *in* $\mathrm{Pred}(X)$ *of predicates* $m\colon\ U \rightarrowtail X$ *and* $n\colon\ V \rightarrowtail Y$ *is the image of the cotuple* $[m, n]\colon U + V \to X$.

Solution.

1. Fix any object $X \in \mathbb{C}$ and any predicate $U \overset{m}{\rightarrowtail} X$. Let $0 \xrightarrow{!_X} X$ and $0 \xrightarrow{!_U} U$ be the unique morphisms from 0 to X and U respectively. Then we have the commuting diagram

$$\begin{tikzcd}
0 \arrow[r, two heads, "\mathfrak{e}(!_X)"] \arrow[d, "!_U"'] & \mathrm{Im}(!_X) \arrow[d, tail, "\mathfrak{m}(!_X)"] \arrow[dl, dashed, "\exists!"'] \\
U \arrow[r, tail, "m"] & X
\end{tikzcd}$$

in $\mathbb{C}$, since $\mathfrak{m}(!_X) \circ \mathfrak{e}(!_X) = !_X = m \circ !_U$ by the initiality of 0.

2. Fix an object $X \in \mathbb{C}$ and predicates $U \overset{m}{\rightarrowtail} X$ and $V \overset{n}{\rightarrowtail} X$. The diagram

$$\begin{tikzcd}
U \arrow[rr, "\kappa_1"] \arrow[dd, tail, "m"'] & & U+V \arrow[dl, two heads, "\mathfrak{e}([m,n])"] \\
& \mathrm{Im}([m,n]) \arrow[dl, tail, "\mathfrak{m}([m,n])"] & \\
X & & V \arrow[ll, tail, "n"] \arrow[uu, "\kappa_2"']
\end{tikzcd}$$

in $\mathbb{C}$ commutes, where κ_1 and κ_2 are the relevant coprojections. So we certainly have the inequalities $m \leq \mathfrak{m}([m,n]) \geq n$ in $\mathrm{Pred}(X)$. Now suppose that there is another predicate $P \overset{p}{\rightarrowtail} X$ satisfying the inequalities $m \leq p \geq n$ in $\mathrm{Pred}(X)$. So there are unique morphisms $U \xrightarrow{f} P \xleftarrow{g} V$ in $\mathbb{C}$ such that the diagram

$$\begin{tikzcd}
& & & P \arrow[dddlll, tail, bend right, "p"'] \\
U \arrow[rr, "\kappa_1"] \arrow[dd, tail, "m"'] \arrow[urrr, bend left, "f"] & & U+V \arrow[ur, "{[f,g]}"'] \arrow[dl, two heads, "\mathfrak{e}([m,n])"] & \\
& \mathrm{Im}([m,n]) \arrow[dl, tail, "\mathfrak{m}([m,n])"] & & \\
X & & V \arrow[ll, tail, "n"] \arrow[uu, "\kappa_2"'] \arrow[uuur, bend right, "g"'] &
\end{tikzcd}$$

in $\mathbb{C}$ commutes. Therefore we have the commuting diagram

$$\begin{tikzcd}
U+V \arrow[r, two heads, "\mathfrak{e}([m,n])"] \arrow[d, "{[f,g]}"'] & \mathrm{Im}([m,n]) \arrow[d, tail, "\mathfrak{m}([m,n])"] \arrow[dl, dashed, "\exists!"'] \\
P \arrow[r, tail, "p"] & X
\end{tikzcd}$$

in $\mathbb{C}$. This yields the inequality $\mathfrak{m}([m,n]) \leq p$ in $\mathrm{Pred}(X)$. $\square$

Exercise 4.3.5

Two morphisms f, g in an arbitary category $\mathbb{C}$ may be called orthogonal, written $f \perp g$, if in each commuting square as below there is a unique diagonal making everything in sight commute:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d] & \bullet \arrow[d] \arrow[dl, dashed] \\
\bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}$$

The diagonal-fill-in property for a factorisation system $(\mathfrak{M}, \mathfrak{E})$ in Definition 4.3.2 thus says that $e \perp m$ for each $m \in \mathfrak{M}$ and $e \in \mathfrak{E}$.

Now assume that a category $\mathbb{C}$ is equipped with a factorisation system $(\mathfrak{M}, \mathfrak{E})$, not necessarily 'logical'. This means that only properties (1)–(3) in Definition 4.3.2 hold.

1. *Prove that $f \in \mathfrak{E}$ if and only if $f \perp m$ for all $m \in \mathfrak{M}$.*
2. *Similarly, prove that $g \in \mathfrak{M}$ if and only if $e \perp g$ for all $e \in \mathfrak{E}$.*
3. *Prove that $e, d \circ e \in \mathfrak{E} \implies d \in \mathfrak{E}$.*
4. *Similarly (or dually), prove $m, m \circ n \in \mathfrak{M} \implies n \in \mathfrak{M}$.*
5. *Prove $m, n \in \mathfrak{M} \implies m \times n \in \mathfrak{M}$, assuming products exist in $\mathbb{C}$.*
6. *Show that diagonals $\Delta = \langle \mathrm{id}, \mathrm{id} \rangle$ are in $\mathfrak{M}$ if and only if all maps in $\mathfrak{E}$ are epis.*

Solution. Recall that a pair $(\mathfrak{M}, \mathfrak{E})$ of collections of morphisms in $\mathbb{C}$ forms a factorisation system on $\mathbb{C}$ if all of the following hold: both $\mathfrak{M}$ and $\mathfrak{E}$ contain all the isomorphisms in $\mathbb{C}$; both $\mathfrak{M}$ and $\mathfrak{E}$ are closed under composition; any morphism in $\mathbb{C}$ can be written as the composition of a morphism in $\mathfrak{E}$ followed by a morphism in $\mathfrak{M}$; and $e \perp m$ for all $e \in \mathfrak{E}$ and $m \in \mathfrak{M}$.

1. The forward direction is simply by definition of a factorisation system. So we are left to prove the converse. Suppose $A \xrightarrow{f} B$ is a morphism in $\mathbb{C}$ with $f \perp m$ for all $m \in \mathfrak{M}$. Then we have the commuting diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathfrak{e}(f)"', two heads] & B \arrow[d, "\mathrm{id}_B"] \arrow[dl, dashed, "\exists! t"'] \\
\mathrm{Im}(f) \arrow[r, "\mathfrak{m}(f)"', tail] & B
\end{tikzcd}$$

in $\mathbb{C}$. In particular, $\mathfrak{m}(f) \circ t = \mathrm{id}_B$. Now, the diagram

$$\begin{tikzcd}
A \arrow[rr, "\mathfrak{e}(f)", two heads] \arrow[dd, "\mathfrak{e}(f)"', two heads] & & \mathrm{Im}(f) \arrow[dl, "\mathfrak{m}(f)"', tail] \arrow[dd, "\mathfrak{m}(f)", tail] \\
& B \arrow[dl, "t"'] & \\
\mathrm{Im}(f) \arrow[rr, "\mathfrak{m}(f)"', tail] & & B
\end{tikzcd}$$

in $\mathbb{C}$ also commutes. By the uniqueness clause in the diagonal-fill-in property for factorisation systems, we must have $t \circ \mathfrak{m}(f) = \mathrm{id}_{\mathrm{Im}(f)}$. Hence $\mathrm{Im}(f) \xrightarrow{\mathfrak{m}(f)} B$ is an isomorphism (with inverse $B \xleftarrow{t} \mathrm{Im}(f)$), and so $\mathfrak{m}(f) \in \mathfrak{E}$. Therefore $f = \mathfrak{m}(f) \circ \mathfrak{e}(f) \in \mathfrak{E}$, since $\mathfrak{E}$ is closed under composition.

2. Again, the forward direction is by definition of a factorisation system. For the converse, suppose $A \xrightarrow{g} B$ is a morphism in $\mathbb{C}$ with $e \perp g$ for all $e \in \mathfrak{E}$. Then the commuting diagram

$$\begin{tikzcd}
A \arrow[r, "\mathfrak{e}(g)"] \arrow[d, "\mathrm{id}_A"'] & \mathrm{Im}(g) \arrow[d, "\mathfrak{m}(g)"] \arrow[dl, dashed, "\exists! t"'] \\
A \arrow[r, "g"] & B
\end{tikzcd}$$

in $\mathbb{C}$ implies that $t \circ \mathfrak{e}(g) = \mathrm{id}_A$. Furthermore, the commuting diagram

$$\begin{tikzcd}
A \arrow[rr, "\mathfrak{e}(g)"] \arrow[dd, "\mathfrak{e}(g)"'] & & \mathrm{Im}(g) \arrow[dl, "t"'] \arrow[dd, "\mathfrak{m}(g)"] \\
& A \arrow[dl, "\mathfrak{e}(g)"'] & \\
\mathrm{Im}(g) \arrow[rr, "\mathfrak{m}(g)"'] & & B
\end{tikzcd}$$

in $\mathbb{C}$ implies that $\mathfrak{e}(g) \circ t = \mathrm{id}_{\mathrm{Im}(g)}$. Thus $\mathfrak{e}(g) \in \mathfrak{M}$ and so $g = \mathfrak{m}(g) \circ \mathfrak{e}(g) \in \mathfrak{M}$.

3. Let $A \xrightarrow{e} B \xrightarrow{d} C$ be morphisms in $\mathbb{C}$ with $e, de \in \mathfrak{E}$. Then the diagram

$$\begin{tikzcd}
A \arrow[r, "de"] \arrow[d, "e"'] & C \arrow[dd, "\mathrm{id}_C"] \arrow[ddl, dashed, "\exists! k"'] \\
B \arrow[d, "\mathfrak{e}(d)"'] & \\
\mathrm{Im}(d) \arrow[r, "\mathfrak{m}(d)"'] & C
\end{tikzcd}$$

in $\mathbb{C}$ commutes, and so $\mathfrak{m}(d) \circ k = \mathrm{id}_C$. The commutativity of the diagram

$$\begin{tikzcd}
A \arrow[rr, "\mathfrak{e}(d) \circ e"] \arrow[dd, "\mathfrak{e}(d) \circ e"'] & & \mathrm{Im}(d) \arrow[dl, "\mathfrak{m}(d)"'] \arrow[dd, "\mathfrak{m}(d)"] \\
& C \arrow[dl, "k"'] & \\
\mathrm{Im}(d) \arrow[rr, "\mathfrak{m}(d)"'] & & C
\end{tikzcd}$$

in $\mathbb{C}$ then implies that $k \circ \mathfrak{m}(d) = \mathrm{id}_{\mathrm{Im}(d)}$. So $\mathfrak{m}(d)$ is an isomorphism, giving us $d \in \mathfrak{E}$.

4. We could proceed similarly as in our solution to part (3), but we will instead present a different proof using the result established in part (2).

   Let $A \xrightarrow{n} B \xrightarrow{m} C$ be morphisms in $\mathbb{C}$ with $m, mn \in \mathfrak{M}$. Fix any $(X \xrightarrow{e} Y) \in \mathfrak{E}$ and morphisms $X \xrightarrow{f} A$ and $Y \xrightarrow{g} B$ in $\mathbb{C}$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "e", -{Triangle[open]}] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
A \arrow[r, "n"'] & B
\end{tikzcd}$$

in $\mathbb{C}$ commutes. The diagonal-fill-in property gives us the commuting diagram

$$\begin{tikzcd}
X \arrow[r, "e", -{Triangle[open]}] \arrow[d, "f"'] & Y \arrow[d, "mg"] \arrow[dl, dashed, "\exists! k"'] \\
A \arrow[r, "mn"', {Triangle[open]}->] & C
\end{tikzcd}$$

in $\mathbb{C}$, that is, there exists a unique morphism $Y \xrightarrow{k} A$ satisfying $ke = f$ and $mnk = mg$. Then we have the two commuting diagrams

$$\begin{tikzcd}
X \arrow[r, "e", -{Triangle[open]}] \arrow[d, "nf"'] & Y \arrow[d, "mg"] \arrow[dl, "nk"'] \\
B \arrow[r, "m"', {Triangle[open]}->] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "e", -{Triangle[open]}] \arrow[d, "nf"'] & Y \arrow[d, "mg"] \arrow[dl, "g"'] \\
B \arrow[r, "m"', {Triangle[open]}->] & C
\end{tikzcd}$$

in $\mathbb{C}$, from which it follows that $nk = g$. Any other morphism $Y \xrightarrow{h} A$ satisfying $he = f$ and $nh = g$ will also satisfy $mnh = mg$, and hence $h = k$.

5. Suppose $\mathbb{C}$ has products. Fix $(A \overset{m}{\rightarrowtail} X), (B \overset{n}{\rightarrowtail} Y) \in \mathfrak{M}$, and suppose that we have morphisms $(P \xrightarrow{e} Q) \in \mathfrak{E}$, $P \xrightarrow{f} A \times B$, and $Q \xrightarrow{g} X \times Y$ such that $ge = (m \times n)f$. So the diagram

$$\begin{tikzcd}
P \arrow[rr, "e", -{Triangle[open]}] \arrow[dd, "f"'] & & Q \arrow[dd, "g"] & \\
& A \arrow[rr, "m", {Triangle[open]}->] & & X \\
A \times B \arrow[ur, "\pi_1"] \arrow[rr, "m \times n"] \arrow[dr, "\pi_2"'] & & X \times Y \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\
& B \arrow[rr, "n"', {Triangle[open]}->] & & Y
\end{tikzcd}$$

in $\mathbb{C}$ commutes, where π_1, π_2, p_1, and p_2 are the relevant projections. This yields the commuting diagram

$$\begin{tikzcd}
P \arrow[r, "e", -{Triangle[open]}] \arrow[d, "f"'] & Q \arrow[d, "g"] \arrow[ddl, dashed, "\exists! k"'] \\
A \times B \arrow[d, "\pi_1"'] & X \times Y \arrow[d, "p_1"] \\
A \arrow[r, "m"', {Triangle[open]}->] & X
\end{tikzcd}$$

in $\mathbb{C}$, since $e \perp m$. Similarly, there exists a unique morphism $Q \xrightarrow{\ell} B$ such that $\ell e = \pi_2 f$ and $n\ell = p_2 g$. The morphisms $A \xleftarrow{k} Q \xrightarrow{\ell} B$ induce a unique morphism $Q \xrightarrow{\langle k,\ell\rangle} A \times B$ satisfying

$$\langle k,\ell\rangle e = \langle ke, \ell e\rangle = \langle \pi_1 f, \pi_2 f\rangle = f \quad \text{and} \quad (m \times n)\langle k,\ell\rangle = \langle mk, n\ell\rangle = \langle p_1 g, p_2 g\rangle = g,$$

yielding $e \perp (m \times n)$. Therefore $(m \times n) \in \mathfrak{M}$, by part (2) of this exercise.

6. Continue assuming that $\mathbb{C}$ has products. Let us start with the forward direction, supposing that $\langle \mathrm{id}_X, \mathrm{id}_X\rangle \in \mathfrak{M}$ for all $X \in \mathbb{C}$. Fix a morphism $(\ A \xrightarrow{e} B\ ) \in \mathfrak{E}$ and suppose we have a parallel pair of morphisms $B \underset{g}{\overset{f}{\rightrightarrows}} C$ in $\mathbb{C}$ satisfying $fe = ge$. Then, the commuting diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "fe=ge"'] & B \arrow[d, "\langle f,g\rangle"] \arrow[dl, dashed, "\exists!h"'] \\ C \arrow[r, "\langle \mathrm{id}_C,\mathrm{id}_C\rangle"'] & C \times C \end{tikzcd}$$

in $\mathbb{C}$ asserts, in particular, that there is a morphism $B \xrightarrow{h} C$ satisfying $\langle h,h\rangle = \langle f,g\rangle$. It follows that $f = g$.

Conversely, suppose that every morphism in $\mathfrak{E}$ is epic. Fix any $X \in \mathbb{C}$ and suppose we have a commuting square of the form

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, "\langle \mathrm{id}_X,\mathrm{id}_X\rangle"'] & X \times X \end{tikzcd}$$

in $\mathbb{C}$, where $e \in \mathfrak{E}$. Letting $X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$ denote the projection morphisms for the binary product $X \times X$, we have $\pi_1 ge = f = \pi_2 ge$. As e is epic, we have $\pi_1 g = \pi_2 g$. We thus obtain the commuting diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, "\pi_1 g=\pi_2 g"'] \\ X \arrow[r, "\langle \mathrm{id}_X,\mathrm{id}_X\rangle"'] & X \times X \end{tikzcd}$$

in $\mathbb{C}$. Any other morphism $B \xrightarrow{h} X$ satisfying $he = f$ and $\langle \mathrm{id}_X, \mathrm{id}_X\rangle h = g$ will, in particular, satisfy $he = f = (\pi_1 g)e$, from which it follows that $h = \pi_1 g$ since e is epic. We conclude that $e \perp \langle \mathrm{id}_X, \mathrm{id}_X\rangle$. Therefore $\langle \mathrm{id}_X, \mathrm{id}_X\rangle \in \mathfrak{M}$, by part (2) of this exercise. ☐

Exercise 4.3.6

Prove the converse of Proposition 4.3.5.4: if $\coprod_f(f^{-1}(n) \wedge m) = n \wedge \coprod_f(m)$ holds for all appropriate f, m, n, then $\mathfrak{E}$ is closed under pullback along maps $m \in \mathfrak{M}$.

Solution. Let $(\mathfrak{M}, \mathfrak{E})$ be a factorisation system (see Exercise 4.3.5) on a category $\mathbb{C}$ satisfying all the properties of a logical factorisation system with the exception of the clause that $\mathfrak{E}$ is closed under pullbacks of morphisms in $\mathfrak{M}$. That is, $(\mathfrak{M}, \mathfrak{E})$ satisfies items (1)–(5) of Definition 4.3.2.

For a morphism $X \xrightarrow{f} Y$ in $\mathbb{C}$, recall that the functor $\coprod_f \colon \mathrm{Pred}(X) \to \mathrm{Pred}(Y)$ is defined on objects by sending a predicate $\left(U \overset{m}{\rightarrowtail} X \right)$ to the predicate $\coprod_f(U) := \mathrm{Im}(fm) \overset{\mathfrak{m}(fm)}{\rightarrowtail} Y$. As stipulated in this exercise, suppose that $\coprod_f(f^{-1}(n) \wedge m) = n \wedge \coprod_f(m)$ for all morphisms $X \xrightarrow{f} Y$ in $\mathbb{C}$ and all predicates $\left(U \overset{m}{\rightarrowtail} X \right), \left(V \overset{n}{\rightarrowtail} Y \right) \in \mathfrak{M}$.

Now suppose we have a cospan
$$\begin{tikzcd} & V \arrow[d, tail, "n"] \\ X \arrow[r, two heads, "e"'] & Y \end{tikzcd}$$
in $\mathbb{C}$, with $e \in \mathfrak{E}$ and $n \in \mathfrak{M}$. Form the pullback

$$\begin{tikzcd} U \arrow[r, "n^{-1}(e)"] \arrow[d, tail, "e^{-1}(n)"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, tail, "n"] \\ X \arrow[r, two heads, "e"'] & Y \end{tikzcd}$$

in $\mathbb{C}$. We wish to show that $n^{-1}(e) \in \mathfrak{E}$.

The diagonal-fill-in property tells us that we have the commuting diagram

$$\begin{tikzcd} U \arrow[r, two heads, "\mathfrak{e}(e \circ e^{-1}(n))"] \arrow[d, "n^{-1}(e)"'] & \mathrm{Im}(e \circ e^{-1}(n)) \arrow[d, tail, "\mathfrak{m}(e \circ e^{-1}(n))"] \arrow[dl, dashed, "\exists ! k"'] \\ V \arrow[r, tail, "n"'] & Y \end{tikzcd}$$

in $\mathbb{C}$. In $\mathrm{Pred}(Y)$, we have the equalities

$$\begin{aligned} \mathfrak{m}(e \circ e^{-1}(n)) &= \coprod_e (e^{-1}(n)) \\ &= \coprod_e (e^{-1}(n) \wedge \mathrm{id}_X) \\ &= n \wedge \coprod_e (\mathrm{id}_X) \\ &= n \wedge \mathfrak{m}(e) \\ &= n, \end{aligned}$$

where the third equality follows from our assumption, and the last equality follows from the fact that $\mathfrak{m}(e) = \mathrm{id}_Y$ (as subobjects) since $e \in \mathfrak{E}$ (see Exercise 4.3.1.2). Therefore the unique induced morphism $\mathrm{Im}(e \circ e^{-1}(n)) \xrightarrow{k} V$ must have be an isomorphism, giving us $n^{-1}(e) = k \circ \mathfrak{e}(e \circ e^{-1}(n)) \in \mathfrak{E}$. $\square$

Exercise 4.3.7

Let $(\mathfrak{M}, \mathfrak{E})$ be a factorisation system on a category $\mathbb{C}$ with finite products $1, \times$. Prove that the category of predicates $\mathrm{Pred}(\mathbb{C})$ also has finite products, via the following constructions:

1. *The identity $\left(1 \rightarrowtail 1 \right)$ on the final object $1 \in \mathbb{C}$ is final in $\mathrm{Pred}(\mathbb{C})$.*

2. *The product of predicates* $(m : U \rightarrowtail X)$ *and* $(n : V \rightarrowtail Y)$ *is the conjunction of the pullbacks* $\pi_1^{-1}(m) \wedge \pi_2^{-1}(n)$*, as a predicate on* $X \times Y$*.*

Solution. Given any predicate $U \overset{m}{\rightarrowtail} X$, the diagram

$$\begin{tikzcd}
U \arrow[r, dashed, "!_U"] \arrow[d, "m"'] & 1 \arrow[d, "\mathrm{id}_1"] \\
X \arrow[r, dashed, "!_X"'] & 1
\end{tikzcd}$$

in $\mathbb{C}$ commutes by 1 being the terminal object in $\mathbb{C}$, where $!_X$ and $!_U$ are the unique morphisms from X to 1 and U to 1 respectively.

Now, given two predicates $(m : U \rightarrowtail X)$ and $(n : V \rightarrowtail Y)$, we form the following three pullbacks in $\mathbb{C}$:

$$\begin{tikzcd}
 & & R \arrow[dl, tail, "r_1"'] \arrow[dr, tail, "r_2"] \arrow[dd, phantom, "\vee", very near start] & & \\
U \arrow[d, tail, "m"'] & P \arrow[l, "p"'] \arrow[dr, tail, "\pi_1^{-1}(m)"'] \arrow[dl, phantom, "\llcorner", very near start] & & Q \arrow[r, "q"] \arrow[dl, tail, "\pi_2^{-1}(n)"] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, tail, "n"] \\
X & & X \times Y \arrow[ll, "\pi_1"] \arrow[rr, "\pi_2"'] & & Y
\end{tikzcd}$$

where $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ are the relevant projections. We will show that $r := \pi_1^{-1}(m) \circ r_1 = \pi_2^{-1}(n) \circ r_2$ is the product of m and n in $\mathrm{Pred}(\mathbb{C})$. Suppose we are given a predicate $S \overset{s}{\rightarrowtail} Z$ and a pair of morphisms $m \xleftarrow{f} s \xrightarrow{g} n$ in $\mathrm{Pred}(\mathbb{C})$, so that we have the commuting diagram

$$\begin{tikzcd}
U \arrow[d, tail, "m"'] & S \arrow[l, dashed, "\exists! \tilde{f}"'] \arrow[r, dashed, "\exists! \tilde{g}"] \arrow[d, tail, "s"] & V \arrow[d, tail, "n"] \\
X & Z \arrow[l, "f"] \arrow[r, "g"'] & Y
\end{tikzcd}$$

in $\mathbb{C}$. Then $Z \xrightarrow{\langle f,g \rangle} X \times Y$ is the unique morphism in $\mathbb{C}$ such that $\pi_1 \langle f, g \rangle = f$ and $\pi_2 \langle f, g \rangle = g$. It remains to check that this morphism $\langle f, g \rangle$ is a morphism from s to r in $\mathrm{Pred}(\mathbb{C})$. The commuting diagram

$$\begin{tikzcd}
U \arrow[d, tail, "m"'] & S \arrow[l, "\tilde{f}"'] \arrow[r, "\tilde{g}"] \arrow[d, "\langle fs, gs \rangle"] & V \arrow[d, tail, "n"] \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}$$

in $\mathbb{C}$ implies that there are unique morphisms $P \xleftarrow{h} S \xrightarrow{k} Q$ in $\mathbb{C}$ satisfying $ph = \tilde{f}$, $qk = \tilde{g}$, and $\pi_1^{-1}(m) \circ h = \langle fs, gs \rangle = \pi_2^{-1}(n) \circ k$. So there exists a unique morphism $S \xrightarrow{\ell} R$ in $\mathbb{C}$ such that $r_1 \ell = h$ and $r_2 \ell = k$. Therefore

$$r \circ \ell = \pi_1^{-1}(m) \circ r_1 \circ \ell = \pi_1^{-1}(m) \circ h = \langle fs, gs \rangle = \langle f, g \rangle \circ s,$$

so that $s \xrightarrow{\langle f,g\rangle} r$ really is a morphism in $\mathrm{Pred}(\mathbb{C})$. $\square$

Exercise 4.3.8

Let $(\mathfrak{M}, \mathfrak{E})$ be a logical factorisation system on a category $\mathbb{C}$ with pullbacks. Prove that $\mathfrak{E}$ is closed under pullbacks along arbitrary maps if and only if the so-called Beck–Chevalley condition holds: for a pullback as on the left, the inequality on the right is an isomorphism:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "k"] \\ Z \arrow[r, "g"'] & W \end{tikzcd} \qquad\qquad \coprod_f h^{-1}(m) \leq g^{-1} \coprod_k (m).$$

Solution. #?? $\square$

4.4 Relation Lifting, Categorically

Exercise 4.4.1

#??

Solution. #?? □

Exercise 4.4.2

Prove that split epis are orthogonal to all monos (where orthogonality is defined in Exercise 4.3.5.

Conclude that $\mathfrak{E} \subseteq$ SplitEpis, for a logical factorisation system $(\mathfrak{M}, \mathfrak{E})$, implies $\mathfrak{E} =$ SplitEpis.

Solution. Let $\mathbb{C}$ be a category, let $A \overset{e}{\twoheadrightarrow} B$ be a split epimorphism in $\mathbb{C}$ with section $A \overset{s}{\leftarrowtail} B$. Suppose we have a monomorphism $X \overset{m}{\rightarrowtail} Y$ in $\mathbb{C}$ and morphisms $A \xrightarrow{f} X$ and $B \xrightarrow{g} Y$ in $\mathbb{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, tail, "m"'] & Y \end{tikzcd}$$

in $\mathbb{C}$ commutes. Then we have a morphism $B \xrightarrow{fs} X$ satisfying $mfse = gese = ge = mf$, and so

$$fse = f \quad \text{and} \quad mfs = ges = g$$

since m is monic.

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[l, tail, bend left, "s"] \\ X \arrow[r, tail, "m"'] & Y \end{tikzcd}$$

Furthermore, if $B \xrightarrow{k} X$ is any other morphism satisfying $mk = g$ (and $ke = f$), then $mk = g = mfs$, and so $k = fs$ since m is monic. Therefore e is orthogonal to m.

Now let $(\mathfrak{M}, \mathfrak{E})$ be a logical factorisation system on $\mathbb{C}$, and suppose that every morphism in $\mathfrak{E}$ is split epic. As every morphism in $\mathfrak{M}$ is monic, the result above shows that every split epimorphism is orthogonal to every morphism in $\mathfrak{M}$. Therefore, by Exercise 4.3.5.1, every split epimorphism is in $\mathfrak{E}$. □

Exercise 4.4.3

#??

Solution. #?? □

Exercise 4.4.4

#??

Solution. #?? □

Exercise 4.4.5

#??

Solution. #?? □

Exercise 4.4.6

#??

Solution. #?? □

4.5 Logical Bisimulations

Exercise 4.5.1

Let F be an endofunctor on a category $\mathbb{C}$ with a logical factorisation system. Assume algebras $a\colon F(X) \to X$ and $b\colon F(Y) \to Y$ and a relation $\langle r_1, r_2\rangle\colon\ R \rightarrowtail X \times Y$. Prove that the pair (a, b) is a $\mathrm{Rel}(F)$-algebra $\mathrm{Rel}(F)(R) \to R$ in $\mathrm{Rel}(\mathbb{C})$ — making R a logical congruence — if and only if the object $R \in \mathbb{C}$ carries an F-algebra structure $c\colon F(R) \to R$ making the r_i algebra homomorphisms in

$$\begin{tikzcd}
F(X) \arrow[d, "a"'] & F(R) \arrow[l, "F(r_1)"'] \arrow[r, "F(r_2)"] \arrow[d, "c"] & F(Y) \arrow[d, "b"] \\
X & R \arrow[l, "r_1"] \arrow[r, "r_2"'] & Y
\end{tikzcd}$$

Check that this algebra c, if it exists, is unique.

Solution. The existence of a (necessarily unique) morphism $\mathrm{Rel}(F)(R) \xrightarrow{t} R$ in $\mathbb{C}$ making the diagram

$$\begin{tikzcd}
FR \arrow[r, ->>, "\mathfrak{e}(\langle Fr_1, Fr_2\rangle)"] \arrow[dr, "\langle Fr_1, Fr_2\rangle"'] & \mathrm{Rel}(F)(R) \arrow[r, "t"] \arrow[d, tail, "\mathfrak{m}(\langle Fr_1, Fr_2\rangle)"] & R \arrow[d, tail, "\langle r_1, r_2\rangle"] \\
& FX \times FY \arrow[r, "a \times b"'] & X \times Y
\end{tikzcd}$$

in $\mathbb{C}$ commute implies that the diagram

$$\begin{tikzcd}
F(X) \arrow[d, "a"'] & F(R) \arrow[l, "Fr_1"'] \arrow[r, "Fr_2"] \arrow[d, "t \circ \mathfrak{e}(\langle Fr_1, Fr_2\rangle)"'] & F(Y) \arrow[d, "b"] \\
X & R \arrow[l, "r_1"] \arrow[r, "r_2"'] & Y
\end{tikzcd}$$

in $\mathbb{C}$ commutes.

Conversely, given a morphism $FR \xrightarrow{c} R$ in $\mathbb{C}$ satisfying $aFr_1 = r_1 c$ and $bFr_2 = r_2 c$, we have the commuting diagram

$$\begin{tikzcd}
& \mathrm{Im}(c) \arrow[dr, tail, bend left, "\mathfrak{m}(c)"] & \\
FR \arrow[ur, ->>, bend left, "\mathfrak{e}(c)"] \arrow[rr, bend left, "c"] \arrow[r, ->>, "\mathfrak{e}(\langle Fr_1, Fr_2\rangle)"] \arrow[dr, "\langle Fr_1, Fr_2\rangle"'] & \mathrm{Rel}(F)(R) \arrow[d, tail, "\mathfrak{m}(\langle Fr_1, Fr_2\rangle)"] & R \arrow[d, tail, "\langle r_1, r_2\rangle"] \\
& FX \times FY \arrow[r, "a \times b"'] & X \times Y
\end{tikzcd}$$

in $\mathbb{C}$. The diagonal-fill-in property yields the commuting diagram

$$\begin{tikzcd} FR \arrow[r, "\mathfrak{e}(\langle Fr_1, Fr_2\rangle)"] \arrow[d, "\mathfrak{e}(c)"'] & \mathrm{Rel}(F)(R) \arrow[d, tail, "\mathfrak{m}(\langle Fr_1, Fr_2\rangle)"] \arrow[ddl, dashed, "\exists! t"'] \\ \mathrm{Im}(c) \arrow[d, tail, "\mathfrak{m}(c)"'] & FX \times FY \arrow[d, "a \times b"] \\ R \arrow[r, tail, "\langle r_1, r_2\rangle"'] & X \times Y \end{tikzcd}$$

in $\mathbb{C}$, giving a morphism $(\mathrm{Rel}(F)(R) \xrightarrow{\mathfrak{m}(\langle Fr_1, Fr_2\rangle)} FX \times FY) \xrightarrow{(a,b)} (R \xrightarrow{\langle r_1, r_2\rangle} X \times Y)$ in $\mathrm{Rel}(\mathbb{C})$.

Finally, such an F-algebra structure $FR \xrightarrow{c} R$, if it exists, must be unique simply because the relation $R \xrightarrow{\langle r_1, r_2\rangle} X \times Y$ is a monomorphism in $\mathbb{C}$. □

Exercise 4.5.2
#??

Solution. #?? □

Exercise 4.5.3
#??

Solution. #?? □

Exercise 4.5.4
#??

Solution. #?? □

Exercise 4.5.5
#??

Solution. #?? □

Exercise 4.5.6
#??

Solution. #?? □

4.6 Existence of Final Coalgebras

Exercise 4.6.1
#??

Solution. #?? □

Exercise 4.6.2
#??

Solution. #?? □

Exercise 4.6.3
#??

Solution. #?? □

Exercise 4.6.4
#??

Solution. #?? □

Exercise 4.6.5
#??

Solution. #?? □

Exercise 4.6.6
#??

Solution. #?? □

Exercise 4.6.7
#??

Solution. #?? □

Exercise 4.6.8
#??

Solution. #?? □

Exercise 4.6.9
#??

Solution. #?? □

4.7 Polynomial and Analytical Functors

Exercise 4.7.1
#??

Solution. #?? ∎

Exercise 4.7.2
#??

Solution. #?? ∎

Exercise 4.7.3
#??

Solution. #?? ∎

5 Monads, Comonads and Distributive Laws

6 Invariants and Assertions

Bibliography and References

Michael Abbott, Thorsten Altenkirch, and Neil Ghani. Categories of containers. In Andrew D. Gordon, editor, *Foundation of Software Science and Computation Structures*, volume 2620 of *Lecture Notes in Computer Science*, pages 23–38. Springer-Verlag Berlin Heidelberg, 2003a. DOI: https://doi.org/10.1007/3-540-36576-1_2.

Michael Abbott, Thorsten Altenkirch, Neil Ghani, and Conor McBridge. Derivatives of containers. In Martin Hofmann, editor, *Typed Lambda Calculi and Applications*, volume 2701 of *Lecture Notes in Computer Science*, pages 16–30. 2003b. DOI: https://doi.org/10.1007/3-540-44904-3_2.

Michael Abbott, Thorstern Alternkirch, and Neil Ghani. Containers: Constructing strictly positive types. *Theoretical Computer Science*, 342:3–27, 2005. DOI: https://doi.org/10.1016/j.tcs.2005.06.002.

Jiří Adámek. Observability and Nerode equivalence in concrete categories. In Ferenc Gécseg, editor, *Fundamentals of Computation Theory*, volume 117 of *Lecture Notes in Computer Science*, pages 1–15. Springer–Verlag Berlin Heidelberg, 1981. DOI: https://doi.org/10.1007/3-540-10854-8_1.

Michael A. Arbib and Ernest G. Manes. Foundations of systems theory: Decomposable systems. *Automatica*, 10:285–302, 1974. DOI: https://doi.org/10.1016/0005-1098(74)90039-9.

Michael A. Arbib and Ernest G. Manes. Adjoint machines, state-behaviour machines, and duality. *Journal of Pure and Applied Algebra*, 6:313–344, 1975a. DOI: https://doi.org/10.1016/0022-4049(75)90028-6.

Michael A. Arbib and Ernest G. Manes. *Arrows, Structures, and Functors: The Categorical Imperative*. Academic Press, 1975b.

Michael A. Arbib and Ernest G. Manes. Foundations of system theory: The Hankel matrix. *Journal of Computer and System Sciences*, 20:330–378, 1980. DOI: https://doi.org/10.1016/0022-0000(80)90012-4.

Michel Bidoit, Rolf Hennicker, and Alexander Kurz. On the duality between observability and reachability. In Furio Honsell and Marino Miculan, editors, *Foundations of Software Science and Computational Structures*, volume 2030 of *Lecture Notes in Computer Science*, pages 72–87. Springer-Verlag Berlin Heidelberg, 2001. DOI: https://doi.org/10.1007/3-540-45315-6_5.

Marcello M. Bonsangue, Jan Rutten, and Alexandra Silva. Coalgebraic logic and synthesis of Mealy machines. In Roberto Amadio, editor, *Foundations of Software Science and Computational Structures*, volume 4962 of *Lecture Notes in Computer Science*. Springer-Verlag Berlin Heidelberg, 2008. DOI: https://doi.org/10.1007/978-3-540-78499-9_17.

Francis Borceux. *Handbook of Categorical Algebra*, volume 50–52 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1994.
DOIs:
Volume 1, https://doi.org/10.1017/CBO9780511525858;

Volume 2, https://doi.org/10.1017/CBO9780511525865;
Volume 3, https://doi.org/10.1017/CBO9780511525872.

J. Robin B. Cockett. Introduction to distributive categories. *Mathematical Structures in Computer Science*, 3:277–307, 1993.
DOI: https://doi.org/10.1017/S0960129500000232.

J. Robin B. Cockett and Dwight Spencer. Strong categorical datatypes I. In Robert A. G. Seely, editor, *International Meeting on Category Theory 1991*, volume 13, pages 141–169. Canadian Mathematical Society Proceedings, AMS, Montreal, 1992.

J. Robin B. Cockett and Dwight Spencer. Strong categorical datatypes II: A term logic for categorical programming. *Theoretical Computer Science*, 139:69–113, 1995.
DOI: https://doi.org/10.1016/0304-3975(94)00099-5.

Dion Coumans and Bart Jacobs. Scalars, monads, and categories. In Chris Heunen, Mehrnoosh Sadrzadeh, and Edward Grefenstette, editors, *Quantum Physics and Linguistics: A Compositional, Diagrammatic Discourse*, pages 182–216. Oxford University Press, 2013.
DOI: https://doi.org/10.1093/acprof:oso/9780199646296.003.0007.

Brian A. Davey and Hilary A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.
DOI: https://doi.org/10.1017/CBO9780511809088.

E. Allen Emerson. Temporal and modal logic. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B: Formal Models and Semantics, pages 995–1072. Elsevier B.V., 1990.
DOI: https://doi.org/10.1016/B978-0-444-88074-1.50021-4.

Peter Freyd. Recursive types reduced to inductive types. In *Fifth Annual IEEE Symposium on Logic in Computer Science*, pages 498–507, Philadelphia, 1990.
DOI: https://doi.org/10.1109/LICS.1990.113772.

Joseph A. Goguen. Realization is universal. *Mathematical Systems Theory*, 6:359–374, 1972.
DOI: https://doi.org/10.1007/BF01843493.

Joseph A. Goguen. Discrete-time machines in closed monoidal categories. I. *Journal of Computer and System Sciences*, 10:1–43, 1975.
DOI: https://doi.org/10.1016/S0022-0000(75)80012-2.

H. Peter Gumma, Jesse Hughes, and Tobias Schröder. Distributivity of categories of coalgebras. *Theoretical Computer Science*, 308:131–143, 2003.
DOI: https://doi.org/10.1016/S0304-3975(02)00582-0.

Helle Hvid Hansen and Jan Rutten. Symbolic synthesis of mealy machines from arithmetic bistream functions. *Scientific Annals of Computer Science*, 20:97–130, 2010.
URL: https://hdl.handle.net/1871/39846.

Helle Hvid Hansen, Clemens Kupke, and Eric Pacuit. Neighbourhood structures: Bisimilarity and basic model theory. *Logical Methods in Computer Science*, 5(2):1–38, 2009.
DOI: https://doi.org/10.2168/LMCS-5(2:2)2009.

Helle Hvid Hansen, Clemens Kupke, and Raul Andres Leal. Strong completeness for iteration-free coalgebraic dynamic logics. In Josep Diaz, Ivan Lanese, and Davide Sangiorgi, editors, *Theoretical Computer Science*, volume 8705 of *Lecture Notes in Computer Science*, pages 281–295. International Federation for Information Processing, 2014.
DOI: https://doi.org/10.1007/978-3-662-44602-7_22.

Claudio Hermida and Bart Jacobs. Structural induction and coinduction in a fibrational setting. *Information and Computation*, 145:107–152, 1998.
DOI: https://doi.org/10.1006/inco.1998.2725.

Bart Jacobs. Objects and classes, co-algebraically. In Burkhard Freitag, Cliff B. Jones, Christian Lengauer, and Hans-Jörg Schek, editors, *Object Orientation with Parallelism and Persistence*, The Kluwer International Series in Engineering and Computer Science, pages 83–103. Kluwer Academic Publishers, 1996.
DOI: https://doi.org/10.1007/978-1-4613-1437-0_5.

Bart Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observation*, volume 59 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2017.
DOI: https://doi.org/10.1017/CBO9781316823187.

Rudolf E. Kálmán, Peter L. Falb, and Michael A. Arbib. *Topics in Mathematical Systems Theory*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., 1969.

Alexander Kurz and Jiří Rosický. Operations and equations for coalgebras. *Mathematical Structures in Computer Science*, 15(1):149–166, 2005.
URL: https://dl.acm.org/doi/abs/10.5555/1047125.1047127.

F. William Lawvere. *Functorial Semantics of Algebraic Theories and Some Algebraic Problems in the Context of Functorial Semantics of Algebraic Theories*. PhD thesis, Columbia University, 1963.
URL: http://www.tac.mta.ca/tac/reprints/articles/5/tr5.pdf.

Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer-Verlag New York, Inc., second edition, 1978.
DOI: https://doi.org/10.1007/978-1-4757-4721-8.

Ieke Moerdijk and Erik Palmgreen. Wellfounded trees in categories. *Annals of Pure and Applied Logic*, 104:189–218, 2000.
DOI: https://doi.org/10.1016/S0168-0072(00)00012-9.

Eugenio Moggi. Notions of computation and monads. *Information and Computation*, 93(1):55–92, 1991.
DOI: https://doi.org/10.1016/0890-5401(91)90052-4.

Richard Montague. Universal grammar. *Theoria*, 36(3):373–398, 1970.
DOI: https://doi.org/10.1111/j.1755-2567.1970.tb00434.x.

Milad Niqui. *Formalising Exact Arithmetic: Representations, Algorithms and Proofs*. PhD thesis, Radboud University Nijmegen, 2004.
URL: https://hdl.handle.net/2066/60666.

Dusko Pavlović and Vaughan Pratt. The continuum as a final coalgebra. *Theoretical Computer Science*, 280:105–122, 2002.
DOI: https://doi.org/10.1016/S0304-3975(01)00022-6.

Jan J. M. M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249: 3–80, 2000.
DOI: https://doi.org/10.1016/S0304-3975(00)00056-6.

Dana Scott. Advice on modal logic. In Karel Lambert, editor, *Philosophical Problems in Logic: Some Recent Developments*, pages 143–173. D. Reidel Publishing Company, Dordrecht, Holland, 1970.
DOI: https://doi.org/10.1007/978-94-010-3272-8_7.

Hendrik Tews. Coalgebras for binary methods: Properties of bisimulations and invariants. *RAIRO — Theoretical Informatics and Applications*, 35(1):83–111, 2001.
DOI: https://doi.org/10.1051/ita:2001110.