

# An introduction to sets and functions

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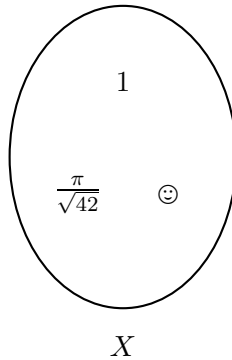
## 0 Blurb

The use of sets and functions is ubiquitous in computer science. They are the bedrock on which all of computer science is built. This piece is meant to serve as an introduction to these concepts. We will see an intuitive description of sets and functions, accompanied by a wealth of examples. We will then study injective, surjective, and bijective functions, which will lead us to a different perspective on the basic concept of counting.

This set of notes was used in a session for students visiting the School of Computer Science at the University of Sheffield. There are a collection of problems included here, intended to challenge the learner. They are to be broken down and solved with guidance. Several of these problems are challenging, especially for those who are not yet familiar with the material.

# 1 Sets

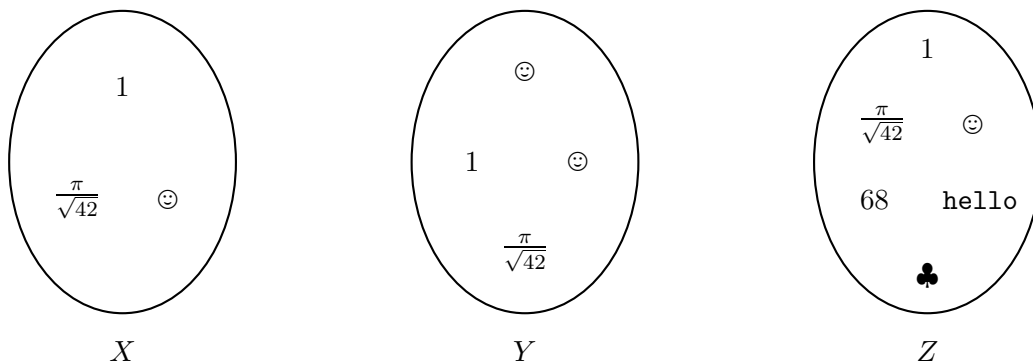
A set is simply a collection of objects, called elements, where membership is the only characteristic we keep track of.<sup>1</sup> For instance, the following depiction of a set, which we shall call  $X$ ,



has three elements inside it: 1,  $\frac{\pi}{\sqrt{42}}$ , and  $\text{☺}$ . We shall use the symbol “ $\in$ ” as a shorthand to help us denote the membership: we write “ $1 \in X$ ” to abbreviate the phrase “1 is an element in the set  $X$ ”. Thus, with the set  $X$  above, we have  $1 \in X$ ,  $\frac{\pi}{\sqrt{42}} \in X$ , and  $\text{☺} \in X$ . This is often shortened to “ $1, \frac{\pi}{\sqrt{42}}, \text{☺} \in X$ ”.

To denote non-membership, we use the symbol “ $\notin$ ”. As an example with the set  $X$  above, we would write “ $\clubsuit \notin X$ ” to express that “ $\clubsuit$  is not in  $X$ ”.

We should stress that membership is the *only* characteristic we keep track of when dealing with sets. We are *not* concerned with whether there are multiple copies of any particular element inside a set. We are also *not* concerned with the order in which our elements are arranged in our set. Thus, for the following three sets



we will say that  $X = Y$ , but  $X \neq Z$  and  $Y \neq Z$ .

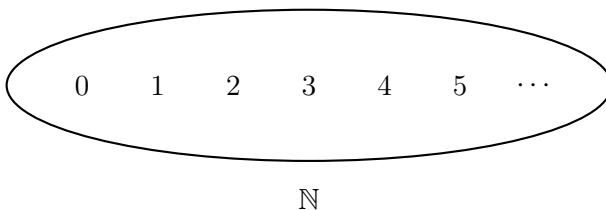
We have made no stipulation that our sets be inhabited. There is a very special set, called the empty set, which has no elements in it. It is often denoted by the symbol “ $\emptyset$ ”. Thus, for any object  $x$ , we have  $x \notin \emptyset$ .



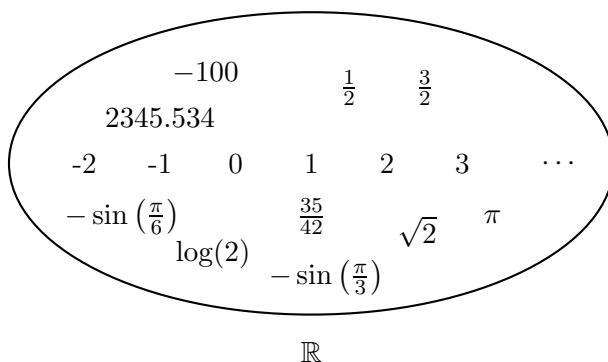
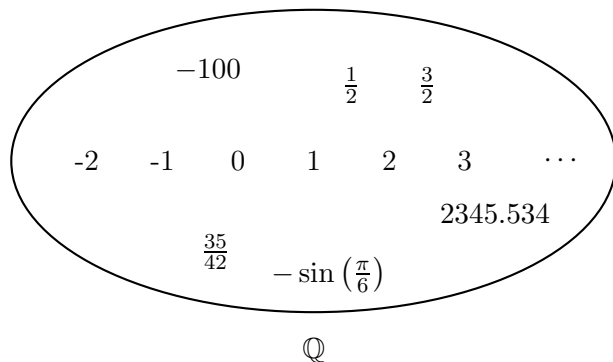
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<sup>1</sup>We are deliberately giving a vague definition of the word “set”, for reasons which we shall not elaborate on.

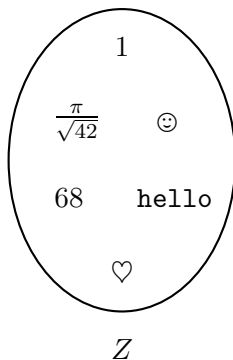
Sets can also be infinite. The set



is commonly called the set of all natural numbers, and is denoted by the symbol “ $\mathbb{N}$ ”. Other popular infinite sets are the set of all rational numbers and the set of all real numbers, denoted by the symbols “ $\mathbb{Q}$ ” and “ $\mathbb{R}$ ” respectively.



It should be evident that drawing sets as ellipses with their elements in them, while visually appealing, is often inconvenient. The modern convention is to instead use the curly braces, “{” and “}”, to encapsulate the elements we wish to be members of our set, separated by commas. As an example, the set



can be more compactly written as

$$Z = \left\{ 1, \frac{\pi}{\sqrt{42}}, \text{☺}, 68, \text{hello}, \heartsuit \right\}.$$

Keeping in mind that we only keep track of membership when talking about sets, we could also write the above set  $Z$  as

$$Z = \left\{ \text{hello}, \text{hello}, 68, 1, \text{☺}, \heartsuit, 1, 1, \frac{\pi}{\sqrt{42}}, \heartsuit \right\}.$$

This notation works for small sets like the set  $Z$  above, but can quickly become inconvenient when dealing with large sets or infinite sets. To build more complicated sets, we introduce **set-builder**

**notation.** If we are given a set  $X$ , we shall write

$$\{x \in X : x \text{ is an even natural number}\}$$

for the set of all elements in  $X$  which are even natural numbers, and nothing else. As a concrete example, if  $X = \{0, 1, 3, 9, \spadesuit, \pi\}$ , then

$$\{x \in X : x \text{ is an even natural number}\} = \{0\}.$$

We can analogously replace “ $x$  is an even natural number” with any property we desire. As another example, if  $Y = \{\text{hello, rbru, blabditgw, university, computer, } \diamond\}$ , then

$$\{y \in Y : y \text{ is an English word}\} = \{\text{hello, university, computer}\}.$$

As a final example,

$$\{p \in \mathbb{N} : p \text{ is a prime number}\} = \{2, 3, 5, 7, 11, 13, 17, \dots\}.$$

**Problem 1**

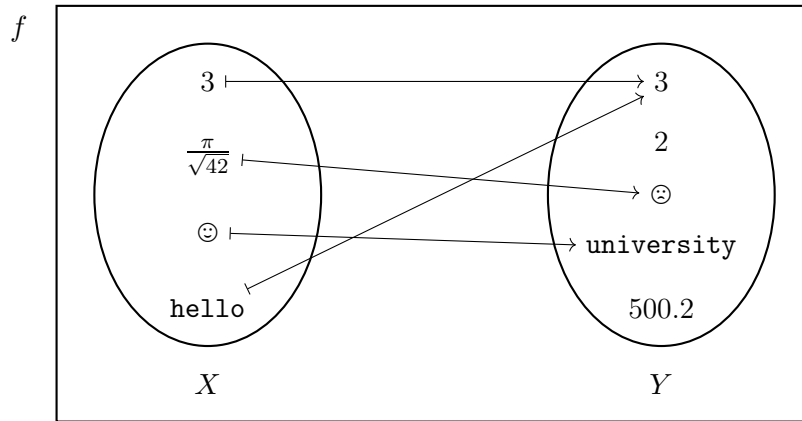
*How many distinct elements are in the set  $\{3, \text{hello}, 4, 0, \diamond, \{3, 2, 1, 3\}, 3, 5, \{\pi\}, \pi, 0, \emptyset\}$ ?*

**Problem 2**

*Let  $\mathbb{N}$  be the set of all natural numbers and let  $\mathbb{R}$  be the set of all real numbers. What are the elements of the set  $\{x \in \mathbb{R} : x \notin \mathbb{N} \text{ or there exists } y \in \mathbb{R} \text{ with } 5y = x\}$ ?*

## 2 Functions

A **function** is an operation between a pair of sets which sends each input element to *exactly* one output element. For instance, the following is a depiction of a function  $f$  from a set  $X$  to a set  $Y$ .

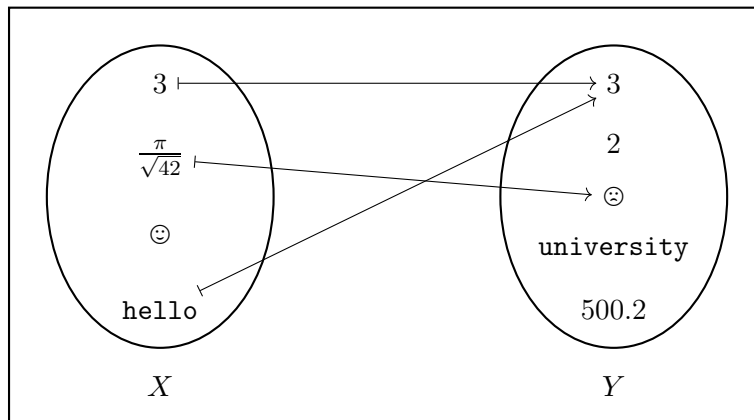


The function  $f$  above takes the set  $X = \left\{3, \frac{\pi}{\sqrt{42}}, \odot, \text{hello}\right\}$  as its set of inputs, and it has the set  $Y = \left\{3, 2, \ominus, \text{university}, 500.2\right\}$  as its set of possible outputs. The function  $f$  above then sends

- 3 to 3,
- $\frac{\pi}{\sqrt{42}}$  to  $\ominus$ ,
- $\odot$  to **university**, and
- **hello** to 3.

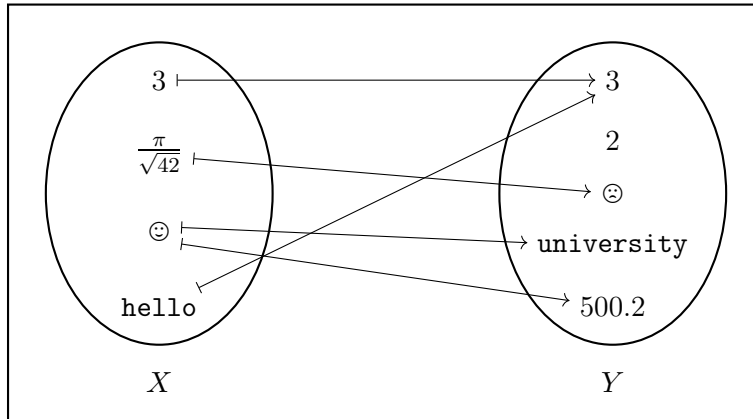
We call the set  $X$  the **domain** of  $f$ , and we call the set  $Y$  the **codomain** of  $f$ . We write “ $f: X \rightarrow Y$ ” to compactly express that “ $f$  is a function with domain  $X$  and codomain  $Y$ ”.

We should stress our stipulation that a function needs to assign to every element in its domain *exactly one* element in its codomain. For instance, the following assignment is *not* a function



because we have not defined the function on the input  $\odot \in X$ . As another example, the following

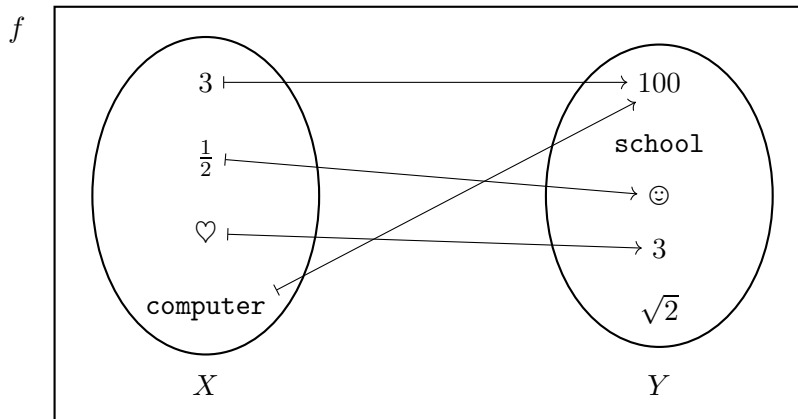
assignment



is *not* a function, because it assigns two outputs, **university** and **500.2**, to the input  $\ominus$ .

Note however, that there is *no* requirement that a function must have each element of its codomain being assigned to some element in its domain. Such functions are rather special, and they will be called **surjective functions**. We will talk more about such surjective functions later.

As with sets, drawing arrows between elements is great for visualising functions, but doing this is often impractical. We will instead adopt the following convention: if  $f: X \rightarrow Y$  is a function (with domain  $X$  and codomain  $Y$ ), then for each  $x \in X$ , we write “ $f(x)$ ” for the element in  $Y$  which  $f$  assigns to  $x$ . For example, with the function  $f: X \rightarrow Y$  depicted below,



we will write

$$\begin{aligned}
 f(3) &= 100, \\
 f\left(\frac{1}{2}\right) &= \ominus, \\
 f(\heartsuit) &= 3, \quad \text{and} \\
 f(\text{computer}) &= 100.
 \end{aligned}$$

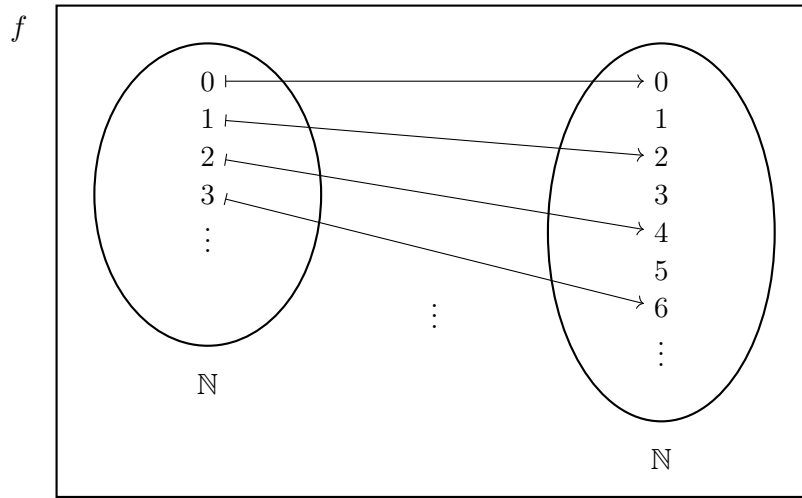
This gives us the power to define functions between really large, or even infinite, sets. For example, we can define a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  by

$$f(n) = 2n, \quad \text{for all } n \in \mathbb{N}.$$

That is, we are defining

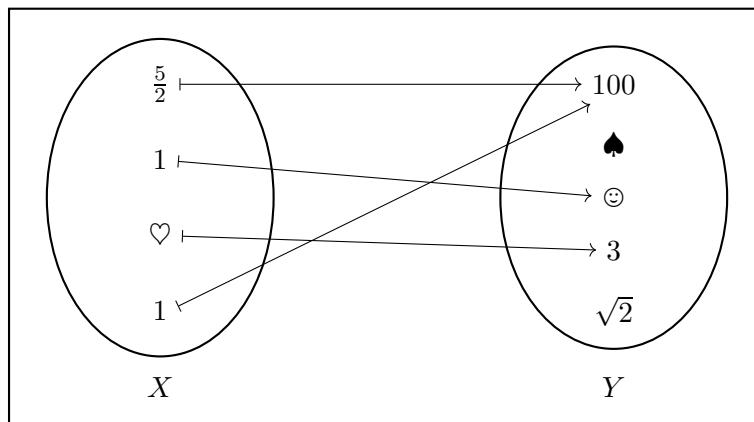
$$f(0) = 0, \quad f(1) = 2, \quad f(2) = 4, \quad f(3) = 6, \quad f(4) = 8, \quad \dots,$$

with the following depiction of the function  $f$ .



**Problem 3**

Consider the following assignment



between the sets  $X = \{\frac{5}{2}, 1, \heartsuit, 1\}$  and  $Y = \{100, \spadesuit, \smiley, 3, \sqrt{2}\}$ . Is this a function?

**Problem 4 (Difficult)**

Let  $\mathbb{N}$  denote the set of all natural numbers. An ordinal is a set  $\alpha$  which is either empty, or is non-empty and satisfies the following four properties:

- every element in  $\alpha$  is a set;
- if  $x$  and  $y$  are such that  $x \in y \in \alpha$ , then  $x \in \alpha$ ;
- for any  $x, y \in \alpha$ , either  $x \in y$ , or  $x = y$ , or  $y \in x$ ;
- there does not exist a function  $f: \mathbb{N} \rightarrow Y$  (with a non-empty codomain  $Y$ ) such that  $f(0) \in \alpha$  and  $f(n + 1) \in f(n)$  for all  $n \in \mathbb{N}$ , that is,

$$\dots \in f(3) \in f(2) \in f(1) \in f(0) \in \alpha.$$

Prove that if  $\alpha$  and  $\beta$  are ordinals, then either  $\alpha \in \beta$ , or  $\alpha = \beta$ , or  $\beta \in \alpha$ .

### 3 Special types of functions

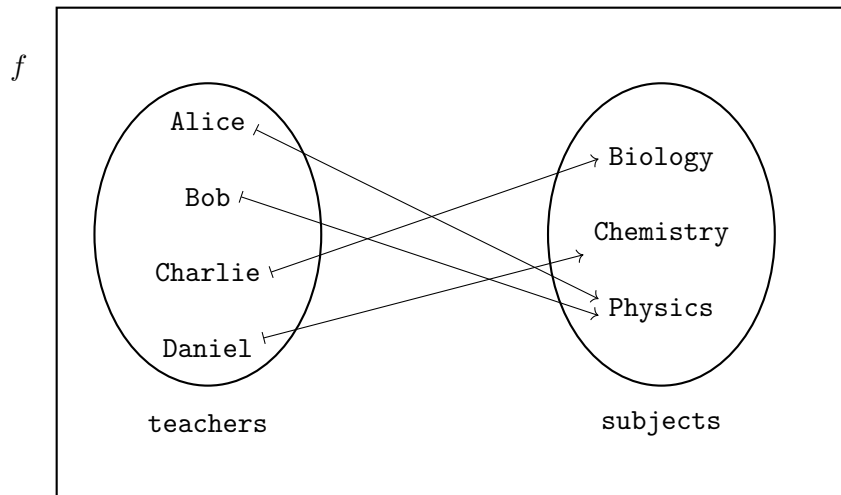
There are certain properties we can ask for from functions which shed some light on those functions. One of them has already been mentioned earlier: the notion of a function which is able to map to every element in its codomain.

#### Definition

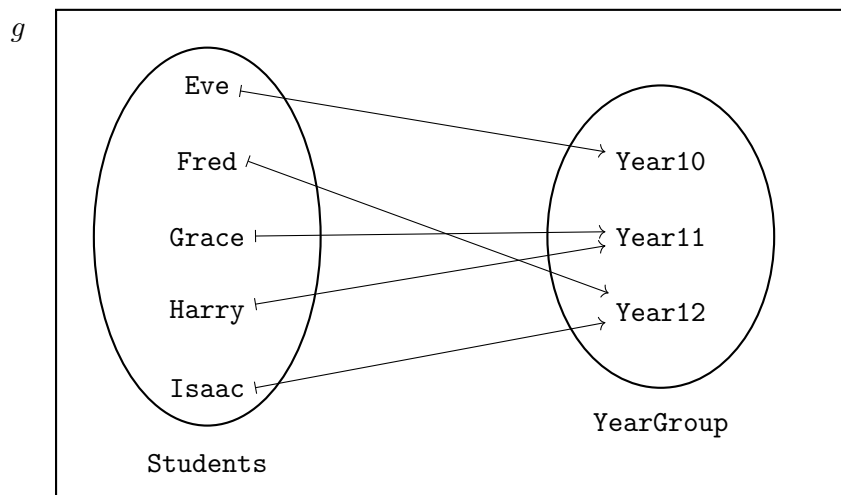
A function  $f: X \rightarrow Y$  is said to be **surjective** precisely when the following property holds:

$$\text{for all } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

For instance, the following function  $f: \text{teacher} \rightarrow \text{subject}$  is surjective.

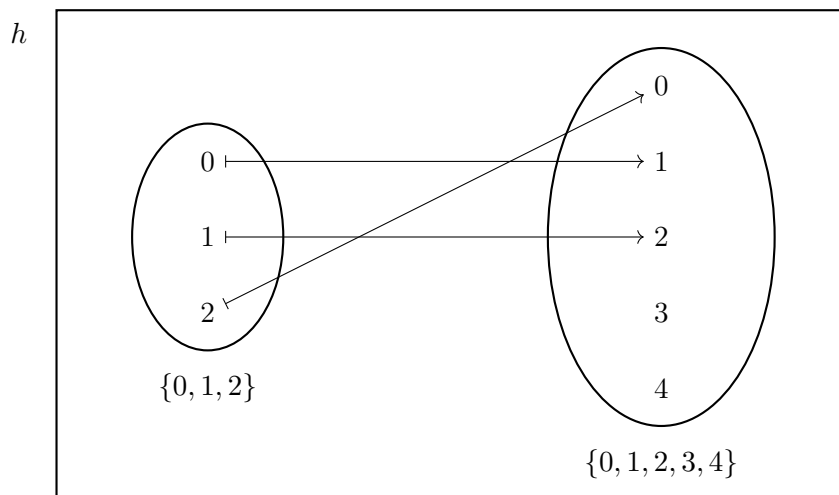


Another example of a surjective function is the following function  $g: \text{Students} \rightarrow \text{YearGroup}$ .



It would also be instructive to see an example of a function which is not surjective. The following

function  $h: \{0, 1, 2\} \rightarrow \{0, 1, 2, 3, 4\}$



is *not* surjective.

**Problem 5**

Let  $\mathbb{N}$  denote the set of all natural numbers and let  $\mathbb{Q}$  denote the set of all rational numbers. Find an example of a function  $f: \mathbb{N} \rightarrow \mathbb{Q}$  which is surjective.

**Problem 6 (Difficult)**

Let  $\mathbb{Q}$  denote the set of rational numbers and let  $\mathbb{R}$  denote the set of real numbers. Prove that there does not exist a surjective function  $f: \mathbb{Q} \rightarrow \mathbb{R}$ .

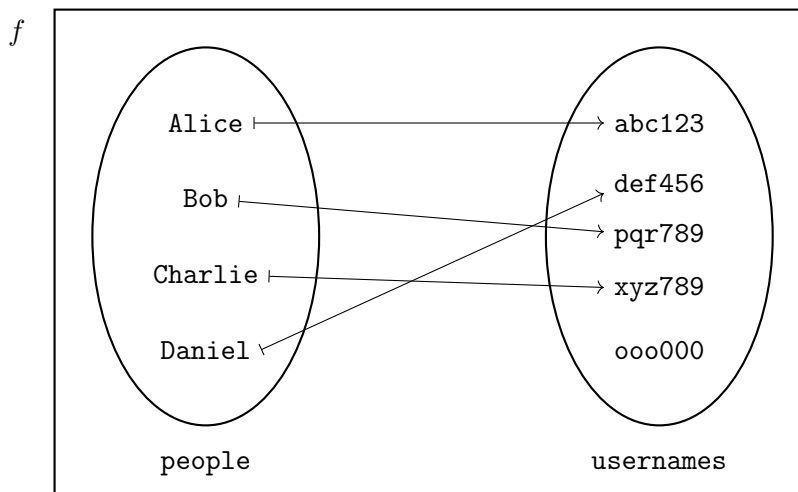
We will now introduce a second special type of function. We notice that functions may often have several elements in their domain mapping to the same element in their codomain. The type of functions we will now introduce will explicitly prohibit this.

**Definition**

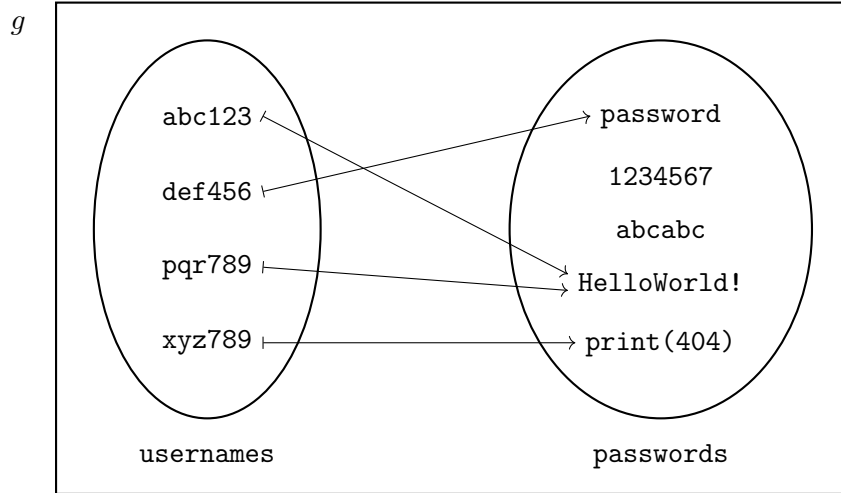
A function  $f: X \rightarrow Y$  is said to be **injective** precisely when the following property holds:

$$\text{for all } x, x' \in X, \text{ if } x \neq x' \text{ then } f(x) \neq f(x').$$

In other words, an injective function sends different elements to different elements. An example depiction of an injective function is the following function  $f: \text{people} \rightarrow \text{usernames}$ ,



whereas the following function  $g: \text{usernames} \rightarrow \text{passwords}$ .



is an example of a function which is *not* injective.

**Problem 7**

Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  be functions such that

$$g(f(x)) = x \quad \text{for all } x \in X.$$

Prove that the function  $f: X \rightarrow Y$  must be injective, and that the function  $g: Y \rightarrow X$  must be surjective.

**Problem 8**

Let  $f: X \rightarrow Y$  be a function. Prove that there exist a set  $Z$ , and two functions  $g: X \rightarrow Z$  and  $h: Z \rightarrow Y$ , such that all of the following three properties hold:

- the function  $g: X \rightarrow Z$  is surjective;
- the function  $h: Z \rightarrow Y$  is injective;
- for any  $x \in X$ , we have  $f(x) = h(g(x))$ .

**Problem 9 (Difficult)**

Fix two functions  $f: X \rightarrow Y$  and  $g: X \rightarrow Y$  with the same domain  $X$  and the same codomain  $Y$ . An equaliser (for  $f$  and  $g$ ) is a function  $e: W \rightarrow X$  such that both of the following hold:

- for all  $w \in W$ , we have  $f(e(w)) = g(e(w))$ ;
- for any  $h: V \rightarrow X$  satisfying

$$f(h(v)) = g(h(v)) \quad \text{for all } v \in V,$$

there exists a unique function  $k: V \rightarrow W$  such that,

$$e(k(v)) = h(v) \quad \text{for all } v \in V.$$

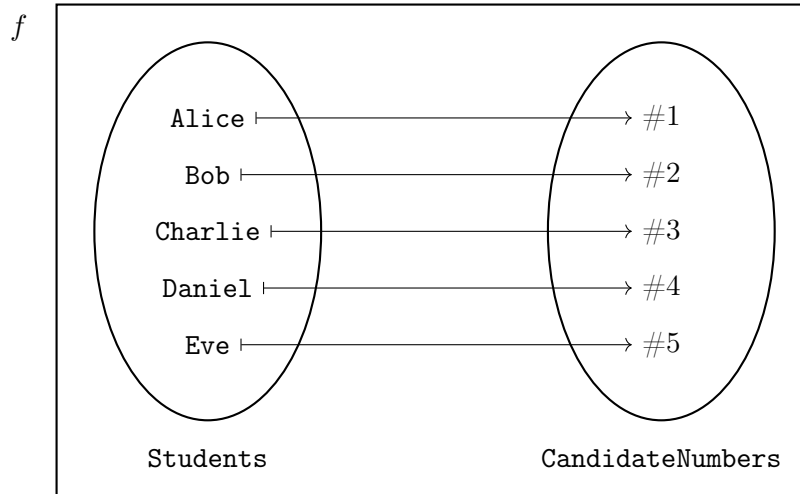
Prove that such an equaliser  $e: W \rightarrow X$  must be injective.

## 4 Counting

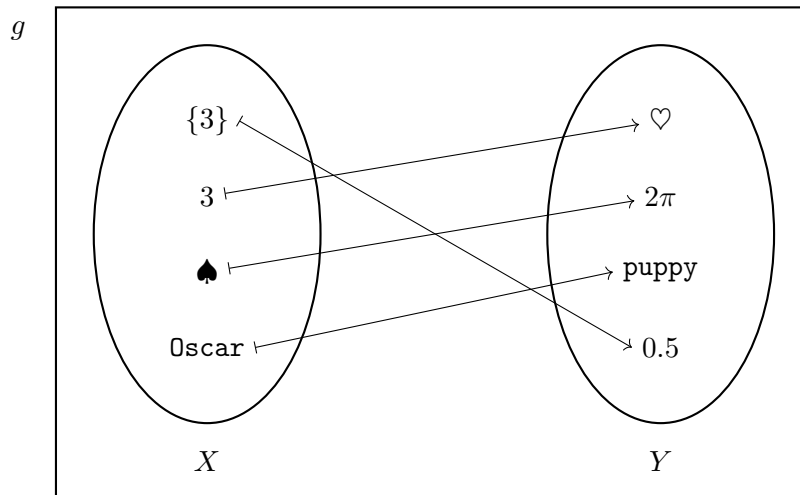
### Definition

A function  $f: X \rightarrow Y$  is said to be **bijective** precisely when  $f$  is both injective and surjective.

That is, a bijective function pairs each element in its domain with a unique element in its codomain, and vice versa. The following function  $f: \text{Students} \rightarrow \text{CandidateNumbers}$



is an example of a bijective function. The following function  $g: X \rightarrow Y$



is another example of a bijective function.

Intuitively, we see that there will exist a bijective function between two finite sets if, and only if, they have the same number of (distinct) elements in them.

### Problem 10

Prove that any function  $f: \{1, 2, 3\} \rightarrow \{1, 2, 3, 4, 5\}$  cannot be bijective.

### Problem 11

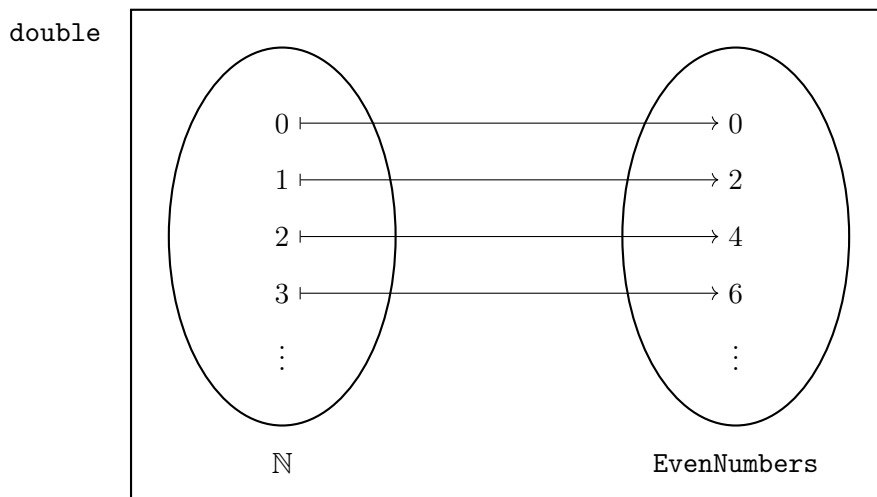
Prove that any function  $g: \{1, 2, 3, 4, 5, 6\} \rightarrow \{1, 2\}$  cannot be bijective.

In fact, this intuition is so on-the-nose that we will adopt it as the *definition* of two sets having the same number of elements in them.

**Definition**

Given two sets  $X$  and  $Y$ , we say that  $X$  and  $Y$  have the same size precisely when there exists a bijective function  $f: X \rightarrow Y$  with domain  $X$  and codomain  $Y$ .

It is important to be aware that this definition carries with it some subtlety. To demonstrate this, consider the following function:  $\text{double}: \mathbb{N} \rightarrow \text{EvenNumbers}$



defined by  $\text{double}(n) = 2n$  for every  $n \in \mathbb{N}$ . This function  $\text{double}: \mathbb{N} \rightarrow \text{EvenNumbers}$  is, in fact, a bijective function. So we would say that  $\mathbb{N}$  and  $\text{EvenNumbers}$  have the same size!

**Problem 12**

Let  $\text{EvenNumbers}$  be the set of all even natural numbers and let  $\mathbb{Q}$  be the set of all rational numbers. Prove that  $\text{EvenNumbers}$  and  $\mathbb{Q}$  have the same size.

We may have intuitively viewed a bijective function  $f: X \rightarrow Y$  as encoding the information that  $X$  and  $Y$  have the same size, and that an injective function  $g: X \rightarrow Y$  as encoding that  $X$  has at most as many elements as  $Y$ . The following important result, known as the Cantor–Bernstein theorem, strengthens this intuitive view. The proof provided below is rather involved, but it demonstrates some important concepts and techniques.

**Theorem**

Let  $X$  and  $Y$  be sets, and suppose we have injective functions  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$ . Then there exists a bijective function  $h: X \rightarrow Y$ .

*Proof.* Let us first establish a couple preliminary definitions. Firstly, a subset of a set  $A$  is a set  $S$  such that

$$\text{for any object } s, \text{ if } s \in S \text{ then } s \in A.$$

We write “ $S \subseteq A$ ” to express that  $S$  is a subset of  $A$ .

Secondly, if we are given a function  $k: A \rightarrow B$ , then we write “ $\{k(a) : a \in A\}$ ” for the image or range of the function  $k$ , i.e. the set of all elements  $b \in B$  for which there exists some  $a \in A$  with  $k(a) = b$ . If we are further given some property  $P(a)$  which can be true or false given an input  $a$ , then we write “ $\{k(a) : a \in A \text{ and } P(a)\}$ ” for the set consisting of all the elements  $b \in B$  for which there exists some  $a \in A$  with  $P(a)$  being true and  $k(a) = b$ .

Now we proceed with the proof. For each subset  $S$  of  $X$ , we define the set

$$S' = \left\{ x \in X : x \notin \left\{ g(y) : y \in Y \text{ and } y \notin \left\{ f(s) : s \in S \right\} \right\} \right\},$$

noting that  $S' \subseteq X$ . Observe that if  $S$  and  $T$  are subsets of  $X$  with  $S \subseteq T$ , then  $S' \subseteq T'$ . Now define the set

$$Z = \{x \in X : \text{there is some subset } S \subseteq X \text{ with } x \in S \subseteq S'\},$$

again noting that  $Z \subseteq X$ . We will argue that  $Z = Z'$ . To show this, we need to show that they have exactly the same elements. That is, we need to show that  $Z \subseteq Z'$  and  $Z' \subseteq Z$ .

For any  $x \in Z$ , there exists some subset  $S \subseteq X$  with  $x \in S \subseteq S'$ . This  $S$  must satisfy  $S \subseteq Z$ , by the definition of  $Z$ , and so  $S' \subseteq Z'$ . This gives us  $x \in S \subseteq S' \subseteq Z'$ , yielding  $x \in Z'$ . So  $Z \subseteq Z'$ . Conversely, as we have established that  $Z \subseteq Z'$ , it follows that  $Z' \subseteq (Z')'$ . So, by the definition of  $Z$ , we must have  $Z' \subseteq Z$ . Therefore  $Z = Z'$ .

Consequently, for any  $x \in X$ , we will have  $x \notin Z$  if, and only if:

$$\text{there exists a unique element } y \in Y \text{ with } y \notin \{f(z) : z \in Z\} \text{ and } g(y) = x.$$

The uniqueness of such an element  $y$  comes from  $g$  being an injective function. So, for each  $x \in X$  with  $x \notin Z$ , we shall write “ $g^{-1}(x)$ ” for the unique element  $y \in Y$  with  $y \notin \{f(z) : z \in Z\}$  and  $g(y) = x$ .

Finally, define a function  $h: X \rightarrow Y$  by

$$h(x) = \begin{cases} f(x), & \text{if } x \in Z, \\ g^{-1}(x), & \text{if } x \notin Z, \end{cases}$$

for every  $x \in X$ . Then  $h: X \rightarrow Y$  is a bijective function:  $h$  is injective because  $f$  is injective and because  $g$  is a function; and  $h$  is surjective because  $Z = Z'$ . □

### Problem 13

Let  $X$  and  $Y$  be sets, and suppose that we have an injective function  $f: X \rightarrow Y$  and a surjective function  $g: X \rightarrow Y$  (note that  $f$  and  $g$  need not be the same function). Prove that there exists a bijective function  $h: X \rightarrow Y$ .